Taiki Saito

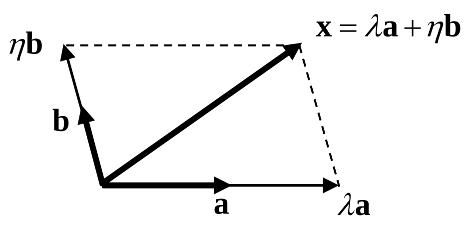
STERA_WAVE Technical Manual Ver.1.0

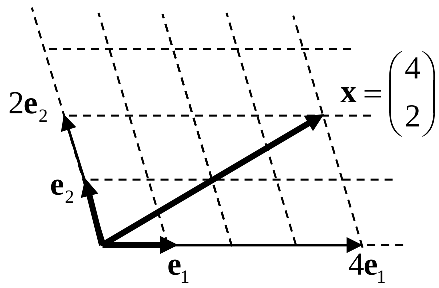
1. Basis of Spectral Analysis

A vector $\mathbf{x}$ in a two-dimensional space ($\mathbf{R}^2$) is represented by the sum of the coefficient times another vectors $\mathbf{a},\mathbf{b}$. If $\mathbf{a},\mathbf{b}$ are linearly independent, the coefficient is uniquely determined.

$$\mathbf{x} = \lambda\mathbf{a} + \eta\mathbf{b} \tag{1}$$

Given vectors $\mathbf{e}_1,\mathbf{e}_2$ of magnitude 1 as vectors $\mathbf{a},\mathbf{b}$, the coefficients are the coordinate components of vector $\mathbf{x}$. In this case, $\mathbf{e}_1,\mathbf{e}_2$ are referred to as basis vectors when $\mathbf{e}_1,\mathbf{e}_2$ are orthogonal, they are referred to as a rectangular coordinate system, and otherwise are referred to as an oblique coordinate system.





When the vector components are $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, the inner product of vectors $\mathbf{a},\mathbf{b}$ is defined as

$$(\mathbf{a},\mathbf{b}) = \mathbf{a}^T\mathbf{b} = a_1b_1 + a_2b_2 = \sum_{i=1}^{2} a_ib_i = |\mathbf{a}||\mathbf{b}|\cos\theta \tag{2}$$

Where, $|\mathbf{a}|$ is the norm of the vector $\mathbf{a}$, and

$$|\mathbf{a}| = \sqrt{(\mathbf{a},\mathbf{a})} = \sqrt{a_1{}^2 + a_2{}^2} \tag{3}$$

θ is the angle between vectors $\mathbf{a},\mathbf{b}$.

For orthogonal basis vectors, the inner product and norm are

$$(\mathbf{e}_1,\mathbf{e}_2) = 0, \quad |\mathbf{e}_1| = \sqrt{(\mathbf{e}_1,\mathbf{e}_1)} = 1, \quad |\mathbf{e}_2| = \sqrt{(\mathbf{e}_2,\mathbf{e}_2)} = 1, \tag{4}$$

When the vector $\mathbf{x}$ is expressed by the orthogonal basis vectors $\mathbf{e}_1, \mathbf{e}_2$ as

$$\mathbf{x} = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \tag{5}$$

Taking inner product between $\mathbf{x}$ and $\mathbf{e}_1, \mathbf{e}_2$, respectively,

$$\lambda_1 = (\mathbf{x}, \mathbf{e}_1), \quad \lambda_2 = (\mathbf{x}, \mathbf{e}_2) \tag{6}$$

They are coordinate components λ_1, λ_2, that is

$$\mathbf{x} = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \tag{7}$$

$$\lambda_i = (\mathbf{x}, \mathbf{e}_i), \quad i = 1, 2 \tag{8}$$

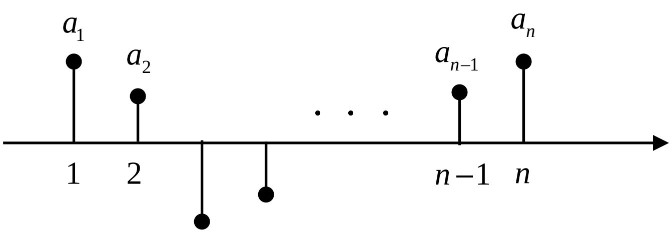
In the same way, the vector $\mathbf{x}$ in the three-dimensional space ($\mathbf{R}^3$) is expressed by the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$\mathbf{x} = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \tag{9}$$

$$\lambda_i = (\mathbf{x}, \mathbf{e}_i), \quad i = 1, 2, 3 \tag{10}$$

and the coordinate components $\lambda_1, \lambda_2, \lambda_3$ in the three-dimensional space are obtained.

Here, the vector departs from the image of "arrow" and generalizes as follows. Now, let the components of the vector $\mathbf{a} \in \mathbf{R}^n$ belonging to the dimensional space ($\mathbf{R}^n$) be $(a_1 \ \ a_2 \ \ \cdots \ \ a_n)^T$. The following figure can be drawn by arranging the components.



For any vectors $\mathbf{a}, \mathbf{b}$, the following conditions are satisfied.

① $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$, then $\mathbf{a} + \mathbf{b} \in \mathbf{R}^n$

② $\mathbf{a} \in \mathbf{R}^n$, then $\lambda \mathbf{a} \in \mathbf{R}^n$ (where, λ is a scalar)

The inner product of vectors $\mathbf{a}, \mathbf{b}$ is defined as

$$(\mathbf{a}, \mathbf{b}) = \mathbf{a}^T \mathbf{b} = \sum_{i=1}^{n} a_i b_i \tag{11}$$

It has the following properties.

① $(\mathbf{a}, \mathbf{b}) \geq 0$ especially when $(\mathbf{a}, \mathbf{a}) = 0$ then $\mathbf{a} = \mathbf{0}$

② $(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a})$

③ $(\mathbf{a}+\mathbf{b}, \mathbf{c}) = (\mathbf{a}, \mathbf{c}) + (\mathbf{b}, \mathbf{c})$

④ $(\lambda \mathbf{a}, \mathbf{b}) = \lambda (\mathbf{a}, \mathbf{b})$

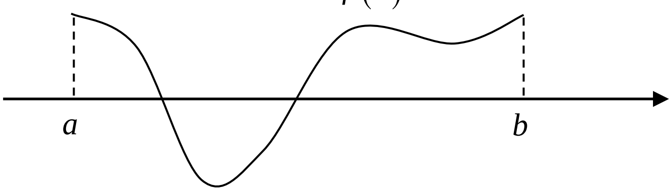
The vector $\mathbf{x} \in \mathbf{R}^n$ is decomposed into components as follows using the orthogonal basis vectors $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n \in \mathbf{R}^n$:

$$\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{e}_i \qquad (12)$$

$$\lambda_i = (\mathbf{x}, \mathbf{e}_i), \quad i = 1, 2, \cdots, n \qquad (13)$$

That is, the coordinate components in the $\mathbf{n}$ dimensional space are obtained.

Next, consider a set of continuous functions $\mathbf{C}$ defined by the interval $[a, b]$ and a function $f(x)$ belonging to $\mathbf{C}$. The function $f(x)$ can be interpreted as a vector with an infinite number of components.



For any functions f, g, if the symbol $\in$ represents that it belongs to the set, then the following conditions are satisfied.

① $f, g \in \mathbf{C}$ then $f + g \in \mathbf{C}$

② $f \in \mathbf{C}$ then $\lambda f \in \mathbf{C}$ (where, λ is a scalar)

This set $\mathbf{C}$ is called function space. The inner product of the functions f, g is defined as

$$(f, g) = \int_a^b f(x) g(x)\, dx \qquad (14)$$

As with vectors, it has the following properties:

① $(f, g) \geq 0$ especially when $(f, f) = 0$ then $f(x) = 0$

② $(f, g) = (g, f)$

③ $(f + g, w) = (f, w) + (g, w)$

④ $(\lambda f, g) = \lambda (f, g)$

As with vectors, when the inner product $(f, g) = 0$, the functions f and g are orthogonal. The norm of the function f is $|f| = \sqrt{(f, f)}$. とする。The function space in which the inner product is defined in this way is called Hilbert space.

The function $f \in \mathbf{C}$ is decomposed into components as follows using a set of mutually orthogonal functions $\phi_1, \phi_2, \cdots, \phi_n \in \mathbf{C}$ of size 1 as

$$f = \sum_{i=1}^{\infty} \lambda_i \phi_i \tag{15}$$

$$\lambda_i = \left(f, \phi_i \right) = \int_a^b f\left(x\right) \phi_i\left(x\right) dx, \quad i = 1, 2, \cdots, n \tag{16}$$

This is called spectral decomposition. That is, in the Hilbert space, a function corresponds to a vector, and a spectrum using an orthogonal function corresponds to a component of orthogonal coordinates.

2. Fourier series expansion

As an example of an orthogonal function sequence $\phi_1, \phi_2, \cdots, \phi_n \in \mathbf{C}$, consider a set of sine and cosine functions.

$$\sin 0\cdot x, \quad \cos 0\cdot x, \quad \sin 1\cdot x, \quad \cos 1\cdot x, \quad \cdots, \quad \sin n\cdot x, \quad \cos n\cdot x, \quad \cdots$$

These are vectors of size 1 that are orthogonal to each other in the interval $(-\pi, \pi)$. The inner product is defined as

$$(f, g) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx \tag{17}$$

Actually

$$(\sin nx, \sin mx) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\cdot\sin mx\,dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{18}$$

$$(\cos nx, \cos mx) = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx\cdot\cos mx\,dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{19}$$

$$(\sin nx, \cos mx) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\cdot\cos mx\,dx = 0 \tag{20}$$

Therefore, from the spectral decomposition of Hilbert space, any function f is decomposed into the following components in the interval $(-\pi, \pi)$.

$$f(x) = \sum_{n=0}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \tag{21}$$

Where

$$\begin{aligned} a_n &= (f, \cos nx) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= (f, \sin nx) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned} \tag{22}$$

In fact, there is one mistake. The following formula

$$(\cos nx, \cos mx) = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx\cdot\cos mx\,dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \tag{23}$$

is wrong when $n = m = 0$, since

$$(\cos 0x, \cos 0x) = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\,dx = 2 \tag{24}$$

Therefore, the coefficient must be half only when $n = 0$. It should be as follows.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \tag{25}$$

It is called Fourier series expansion.

Now we consider the following relationship

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin nx = \frac{e^{inx} - e^{-inx}}{2i} \tag{26}$$

The Fourier series expansion will be

$$f(x) == \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2} e^{inx}\right) + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2} e^{-inx}\right) \tag{27}$$

The first term is, from $b_0 = 0$,

$$\frac{a_0}{2} = \left(\frac{a_n - ib_n}{2} e^{inx}\right)_{n=0} \tag{28}$$

The third term is, from $a_{-n} = a_n, \quad b_{-n} = -b_n$,

$$\sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2} e^{-inx}\right) = \sum_{n=-\infty}^{-1}\left(\frac{a_n - ib_n}{2} e^{inx}\right) \tag{29}$$

Then the Fourier series expansion will be

$$f(x) = \sum_{n=-\infty}^{\infty}\left(\frac{a_n - ib_n}{2} e^{inx}\right) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \tag{30}$$

Where

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \tag{31}$$

(Example)

Obtain Fourier series expansion of $y = x^2$ in the interval $(-\pi \le x \le \pi)$.

(Answer)

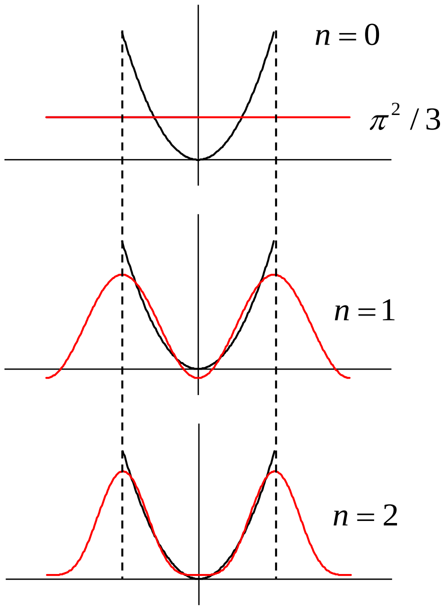
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx = (-1)^n \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx dx = 0$$

Therefore,

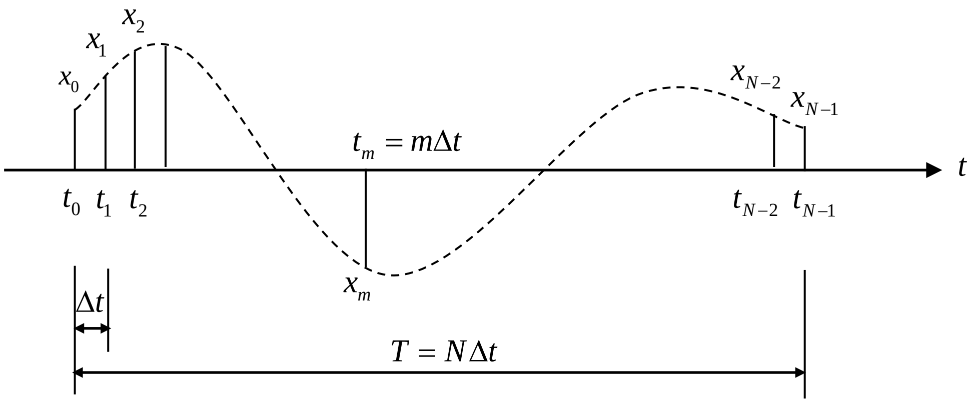
$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos nx$$



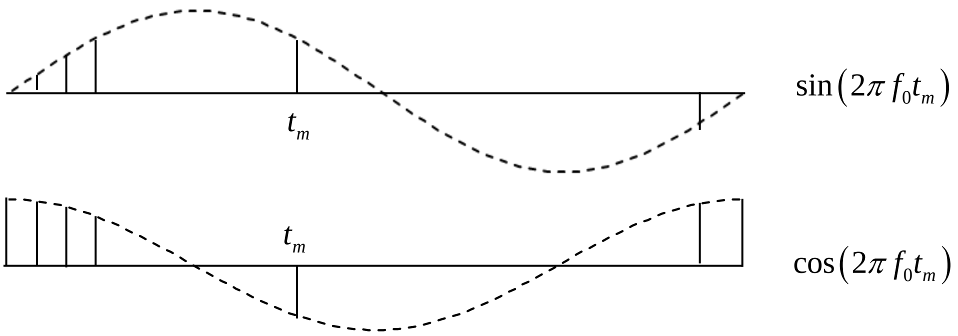
From the figure, it can be seen that the accuracy of approximation increases as the number of terms to be added (n) increases. On the other hand, the Fourier series expansion is an approximation in the interval $(-\pi \le x \le \pi)$, and in other intervals, it is a function that repeats with the period 2π.

3. Finite Fourier series

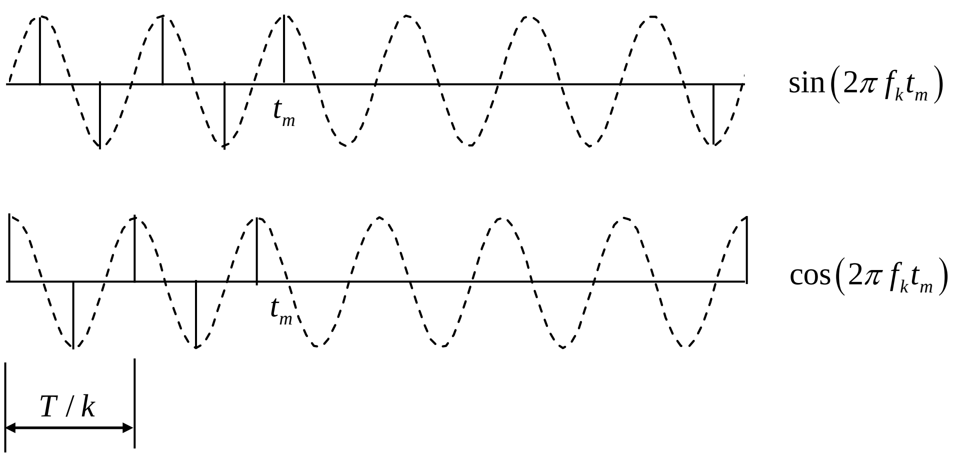
Next, let us consider Fourier series expansion for waveform data composed of discrete values, such as ground motion acceleration data. The waveform is N discrete data $\{x_0, x_1, x_2, \cdots, x_{N-1}\}$ sampled at regular time intervals Δt. The duration of the waveform is $T = N\Delta t$.



The discrete data of a trigonometric function whose period is equal to T will be as follows, where the basic vibration frequency is $f_0 = 1/T$.



When the frequency is $f_k = k\,f_0$



Where

$$2\pi f_k t_m = 2\pi\left(kf_0\right)\left(m\Delta t\right) = \frac{2\pi km}{T/\Delta t} = \frac{2\pi km}{N} \tag{32}$$

Then

$$\sin\left(2\pi f_k t_m\right) = \sin\left(\frac{2\pi km}{N}\right) \quad \text{and} \quad \cos\left(2\pi f_k t_m\right) = \cos\left(\frac{2\pi km}{N}\right) \tag{33}$$

We define the inner product of discrete functions $f(m), g(m), \quad m = 0,1,\cdots,N-1$ as

$$\left(f(m), g(m)\right) = \sum_{m=0}^{N-1} f(m)g(m) \tag{34}$$

Then

$$\left(\sin\left(\frac{2\pi lm}{N}\right), \sin\left(\frac{2\pi km}{N}\right)\right) = \sum_{m=0}^{N-1} \sin\left(\frac{2\pi lm}{N}\right)\sin\left(\frac{2\pi km}{N}\right) = \begin{cases} 0 & l \neq k \\ N/2 & l = k \neq 0 \\ 0 & l = k = 0 \end{cases} \tag{35}$$

$$\left(\cos\left(\frac{2\pi lm}{N}\right), \cos\left(\frac{2\pi km}{N}\right)\right) = \sum_{m=0}^{N-1} \cos\left(\frac{2\pi lm}{N}\right)\cos\left(\frac{2\pi km}{N}\right) = \begin{cases} 0 & l \neq k \\ N/2 & l = k \neq 0 \\ N & l = k = 0 \end{cases} \tag{36}$$

$$\left(\sin\left(\frac{2\pi lm}{N}\right), \cos\left(\frac{2\pi km}{N}\right)\right) = \sum_{m=0}^{N-1} \sin\left(\frac{2\pi lm}{N}\right)\cos\left(\frac{2\pi km}{N}\right) = 0 \tag{37}$$

It can be seen that the discrete data of the trigonometric function is an orthogonal function. Thus, the discrete waveform data $x_m \left(m = 0,1,2,\cdots,N-1\right)$ can be decomposed into components using the orthogonal function. Summing to $N/2$ instead of infinity, it can be expressed as

$$x_m = \sum_{k=0}^{N/2}\left(A_k \cos\frac{2\pi km}{N} + B_k \sin\frac{2\pi km}{N}\right) \tag{38}$$

In the case of $k = 0$, A_0 remains since $\cos\frac{2\pi km}{N} = 1, \ \sin\frac{2\pi km}{N} = 0$. We write it $A_0/2$ so that it fits the expression of the inner product later. Also, in the case of $k = N/2$, $A_{N/2}\cos\frac{2\pi km}{N}$ remains since $\sin\frac{2\pi km}{N} = 0$. We write $A_{N/2}$ as $A_{N/2}/2$.

Then, the series sum is as follows.

$$x_m = \frac{A_0}{2} + \sum_{k=1}^{N/2-1}\left(A_k \cos\frac{2\pi km}{N} + B_k \sin\frac{2\pi km}{N}\right) + \frac{A_{N/2}}{2}\cos\frac{\pi(N/2)m}{N} \tag{39}$$

It is called **Finite Fourier series**. The Fourier coefficient can be obtained from the following equation by using the inner product and the function orthogonality.

$$A_k = \frac{2}{N}\left(x_m, \cos\frac{2\pi km}{N}\right) = \frac{2}{N}\sum_{m=0}^{N-1} x_m \cos\frac{2\pi km}{N}, \quad k = 0,1,2,\cdots,\frac{N}{2} \tag{40}$$

$$B_k = \frac{2}{N}\left(x_m, \sin\frac{2\pi km}{N}\right) = \frac{2}{N}\sum_{m=0}^{N-1} x_m \sin\frac{2\pi km}{N}, \quad k = 0,1,2,\cdots,\frac{N}{2}-1 \tag{41}$$

From $\frac{2\pi km}{N} = \frac{2\pi km}{T/\Delta t} = 2\pi(kf_0)(m\Delta t) = 2\pi f_k t_m$, it can be expressed as

$$x_m = \frac{A_0}{2} + \sum_{k=1}^{N/2-1}\left(A_k \cos(2\pi f_k t_m) + B_k \sin(2\pi f_k t_m)\right) + \frac{A_{N/2}}{2}\cos(2\pi f_{N/2} t_m) \tag{42}$$

Continuous function $x(t)$

$$x(t) \approx \frac{A_0}{2} + \sum_{k=1}^{N/2-1}\left(A_k \cos(2\pi f_k t) + B_k \sin(2\pi f_k t)\right) + \frac{A_{N/2}}{2}\cos(2\pi f_{N/2} t) \tag{43}$$

matches discrete data $x_m\ (m = 0,1,2,\cdots,N-1)$ at N data points $t = t_0, t_1, t_2, \cdots, t_{N-1}$.

The highest frequency is

$$f_{N/2} = \frac{N}{2} f_0 = \frac{N}{2T} = \frac{1}{2\Delta t} \tag{44}$$

This is called the **Nyquist frequency**. In other words, the frequency resolution is represented in the sense that the waveform cannot be represented by a finite Fourier series even if the waveform contains more frequencies. Also, the Fourier series expansion is an approximation in the interval $(0 \le t \le T)$, and in other intervals it is a function that repeats with the period T.

Substituting following relationship to Equation (39)

$$\cos\frac{2\pi km}{N} = \frac{e^{i\frac{2\pi km}{N}} + e^{-i\frac{2\pi km}{N}}}{2}, \quad \sin\frac{2\pi km}{N} = \frac{e^{i\frac{2\pi km}{N}} - e^{-i\frac{2\pi km}{N}}}{2i} \tag{45}$$

Then, the Finite Fourier series will be

$$x_m = \sum_{k=0}^{N/2} \left(\frac{A_k - iB_k}{2} e^{i\frac{2\pi km}{N}} \right) + \sum_{k=1}^{N/2-1} \left(\frac{A_k + iB_k}{2} e^{-i\frac{2\pi km}{N}} \right)$$

Where

$$A_k + iB_k = A_{N-k} - iB_{N-k} \tag{46}$$

Considering

$$e^{-i\left(\frac{2\pi km}{N}\right)} = e^{i\left(\frac{2\pi(N-k)m}{N}\right)} \tag{47}$$

Then, we can get the **Finite complex Fourier series** as follows.

$$x_m = \sum_{k=0}^{N-1} C_k e^{i\left(\frac{2\pi km}{N}\right)}, \quad m = 0,1,2,\cdots,N-1 \tag{48}$$

$$C_k = \frac{A_k - iB_k}{2} = \frac{1}{N} \sum_{m=0}^{N-1} x_m e^{-i\left(\frac{2\pi km}{N}\right)}, \quad k = 0,1,2,\cdots,N-1 \tag{49}$$

Since $C_{N-k} = C_k^*$ (* is conjugate complex number), we need $C_k \left(k = 0,1,2,\cdots,N/2\right)$.

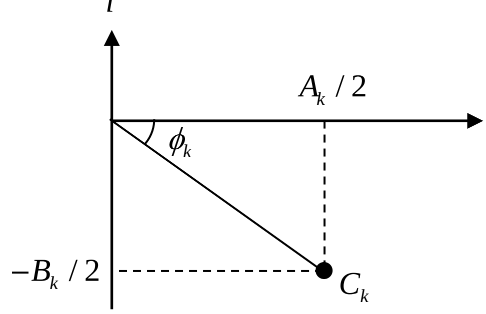
Expressing the Fourier coefficients C_k using amplitude and phase,

$$C_k = \left|C_k\right| e^{i\phi_k}$$

Where

$$\left|C_k\right| = \frac{1}{2}\sqrt{A_k^2 + B_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{A_k}{B_k}\right) = \tan^{-1}\left(\frac{\mathrm{Re}(C_k)}{\mathrm{Im}(C_k)}\right)$$



Note that the graphs of $\left|C_k\right|$ and ϕ_k plotted on the horizontal axis with the frequency are called the amplitude spectrum and the phase spectrum, respectively. However, as described later, the Fourier spectrum of a waveform refers to a value obtained by multiplying the amplitude spectrum $\left|C_k\right|$ by the duration T.

The computer uses this Finite Fourier series to calculate the Fourier transform of the seismic wave. Although not described in this document, an innovative algorithm for obtaining finite Fourier coefficients (Fast Fourier Transform, FFT) is used for calculation.

We can set

$$x_m = x(t), \quad t = m\Delta t \; (m = 0, 1, 2, \cdots, N-1),$$

Duration $T = N\Delta t$, $\Delta\omega = \frac{2\pi}{T} = \frac{2\pi}{N\Delta t}$, $\omega_k = k\Delta\omega \left(k = 0, 1, 2, \cdots, N-1\right)$

Then

$$2\pi km / \mathrm{N} = 2\pi k\left(\frac{t}{\Delta t}\right) / \mathrm{N} = k\left(\frac{2\pi}{T}\right)t = k\Delta\omega t = \omega_k t$$

Therefore, the Finite Fourier transform of $x(t), \quad t = m\Delta t \; (m = 0, 1, 2, \cdots, N-1)$ is

$$C_k = \frac{1}{N}\sum_{m=0}^{N-1} x(t) e^{-i\omega_k t}, \quad k = 0, 1, 2, \cdots, N-1 \tag{50}$$

Its Inverse Finite Fourier transform is

$$x(t) = \sum_{k=0}^{N-1} C_k e^{i\omega_k t}, \quad m = 0, 1, 2, \cdots, N-1 \tag{51}$$

4. Fourier transform

The Fourier transform and inverse Fourier transform of the function $x(t)$ are

$$F(\omega)=\int_{-\infty}^{\infty}x(t)e^{-i\omega t}dt \tag{52}$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}d\omega \tag{53}$$

Using relationship $\omega=2\pi f$, it can be written also

$$F(f)=\int_{-\infty}^{\infty}x(t)e^{-i(2\pi ft)}dt \tag{54}$$

$$x(t)=\int_{-\infty}^{\infty}F(f)e^{i(2\pi ft)}df \tag{55}$$

In the previous Finite Fourier coefficient

$$C_k=\frac{1}{N}\sum_{m=0}^{N-1}x_m e^{-i\left(\frac{2\pi km}{N}\right)},\quad k=0,1,2,\cdots,N-1 \tag{56}$$

Changing the origin of time to be the center of duration $T=N\Delta t$,

$$C_k=\frac{1}{N}\sum_{m=-N/2+1}^{N/2}x_m e^{-i\left(\frac{2\pi km}{N}\right)}=\frac{1}{N\Delta t}\sum_{m=-N/2+1}^{N/2}\left(x_m\Delta t\right)e^{-i\left(\frac{2\pi km}{N}\right)} \tag{57}$$

Increasing the sample number by keeping the duration $T=N\Delta t$ to be constant, by $N\to\infty,\Delta t\to 0$

$$C_k=\frac{1}{T}\int_{-T/2}^{T/2}x(t)e^{-i\left(\frac{2\pi kt}{T}\right)}dt \tag{58}$$

Similarly, the finite Fourier series is given by centering on the origin of the frequency axis.

$$x_m=\sum_{k=-N/2+1}^{N/2}C_k e^{i\left(\frac{2\pi km}{N}\right)},\quad m=-N/2+1,\cdots,0,1,\cdots,N/2 \tag{59}$$

Increasing the sample number by $N\to\infty,\Delta t\to 0$

$$x(t)=\sum_{k=-\infty}^{\infty}C_k e^{i\left(\frac{2\pi kt}{T}\right)} \tag{60}$$

It can be written as

$$x(t)=\sum_{k=-\infty}^{\infty}\left(TC_k\right)e^{i\left(\frac{2\pi kt}{T}\right)}\frac{1}{T} \tag{61}$$

Increasing duration $T \to \infty$, then expressing $\frac{k}{T} \to f, \frac{1}{T} \to df$ and $TC_k \to F(f)$

$$x(t) = \int_{-\infty}^{\infty} F(f) e^{i(2\pi ft)} df \tag{62}$$

It matches the inverse Fourier transform. Similarly, the finite Fourier coefficients are

$$TC_k = \int_{-T/2}^{T/2} x(t) e^{-i\left(\frac{2\pi kt}{T}\right)} dt \tag{63}$$

Increasing duration $T \to \infty$, then expressing $TC_k \to F(f)$

$$F(f) = \int_{-\infty}^{\infty} x(t) e^{-i(2\pi ft)} dt \tag{64}$$

It matches the Fourier transform.

The Fourier spectrum is

$$|F(f)| = |TC_k| = \frac{T}{2}\sqrt{A_k^2 + B_k^2} \tag{65}$$

The Fourier phase spectrum is

$$\phi_k = \tan^{-1}\left(-\frac{A_k}{B_k}\right) = \tan^{-1}\left(\frac{\mathrm{Re}(C_k)}{\mathrm{Im}(C_k)}\right) \tag{66}$$

(Example 1)

Obtain Fourier spectrum of the function

$$x_m = x(t_m) = \sin(\omega_0 t_m) = \sin(2\pi k_0 m / N), \quad t_m = m\Delta t \ (m = 0, 1, 2, \cdots, N-1)$$

(Answer)

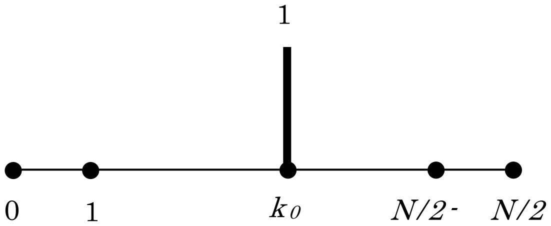
Comparison with Finite Fourier series

$$x_m = \frac{A_0}{2} + \sum_{k=1}^{N/2-1}\left[A_k \cos\frac{2\pi km}{N} + B_k \sin\frac{2\pi km}{N}\right] + \frac{A_{N/2}}{2}\cos\frac{2\pi(N/2)m}{N}$$

The coefficients are

$$A_i = 1, \quad i = k_0, \qquad A_i = 0, \quad i \neq k_0, \qquad B_i = 0$$

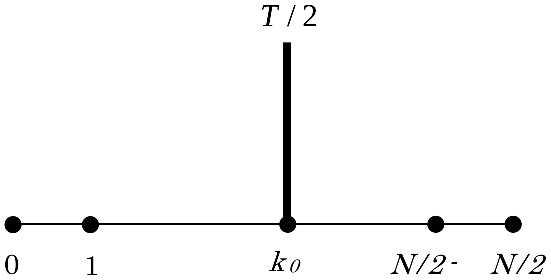
Therefore, the graph of Fourier coefficient is



Fourier spectrum is

$$\left|F(f)\right| = \left|TC_k\right| = \frac{T}{2}\sqrt{A_k^2 + B_k^2} = \frac{T}{2}A_{k_0} = \frac{T}{2} = \frac{N\Delta t}{2}$$

Therefore, the spectrum amplitude is half of duration.



That is, the amplitude of the Fourier spectrum varies depending on the duration even for the same sine waveform.

5. Velocity response spectrum and Fourier spectrum

The Fourier transform of earthquake ground acceleration wave $a(t)$ with duration T is

$$F(\omega)=\int_{-\infty}^{\infty} a(t)e^{-i\omega t}dt=\int_0^T a(t)e^{-i\omega t}dt=\int_0^T a(t)\cos\omega t\,dt-i\int_0^T a(t)\sin\omega t\,dt \tag{67}$$

Then, the Fourier spectrum is

$$\left|F(\omega)\right|=\sqrt{\left(\int_0^T a(t)\cos\omega t\,dt\right)^2+\left(\int_0^T a(t)\sin\omega t\,dt\right)^2} \tag{68}$$

On the other hand, the equation of motion of a single degree of freedom system with input $a(t)$ is

$$m\ddot{x}+c\dot{x}+kx=-ma \tag{69}$$

or

$$\ddot{x}+2h\omega\dot{x}+\omega^2x=-a \tag{70}$$

The displacement $x(t)$ is calculated by

$$x(t)=-\frac{1}{\omega'}\int_0^t a(\tau)e^{-h\omega(t-\tau)}\sin\omega'(t-\tau)d\tau,\quad \omega'=\omega\sqrt{1-h^2} \tag{71}$$

The velocity response is

$$\dot{x}(t)=\int_0^t a(\tau)e^{-h\omega(t-\tau)}\cos\omega'(t-\tau)d\tau-\frac{h}{\sqrt{1-h^2}}\int_0^t a(\tau)e^{-h\omega(t-\tau)}\sin\omega'(t-\tau)d\tau \tag{72}$$

When the damping factor $h=0$,

$$\dot{x}(t)=\int_0^t a(\tau)\cos\omega(t-\tau)d\tau=A\cos\omega t+B\sin\omega t=\sqrt{A^2+B^2}\cos(t-\theta) \tag{73}$$

Where

$$A=\int_0^t a(\tau)\cos\omega\tau d\tau,\quad B=\int_0^t a(\tau)\sin\omega\tau d\tau,\quad \tan\theta=A/B \tag{74}$$

Therefore, the maximum value of velocity response, i.e., response velocity response $S_V(\omega)$) is calculated by

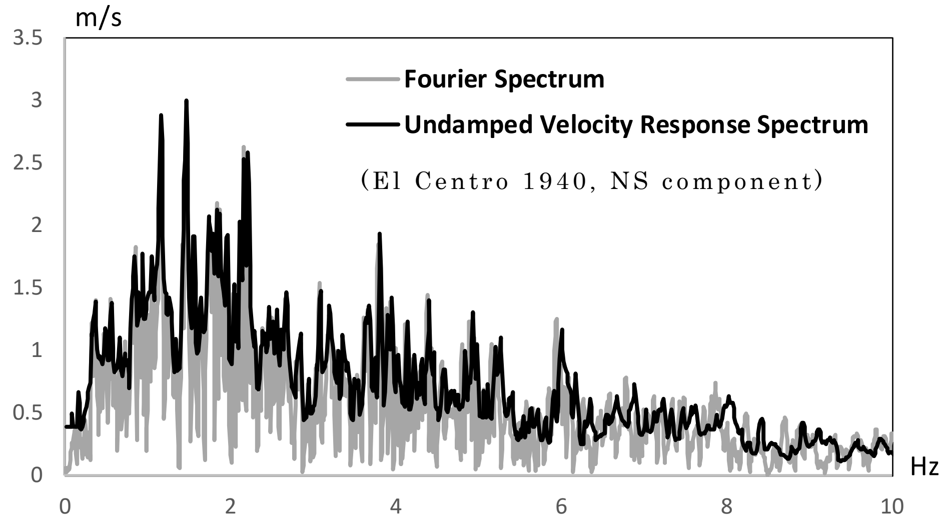
$$S_V(\omega)=\left|\sqrt{A^2+B^2}\right|_{\max}=\left|\sqrt{\left(\int_0^t a(t)\cos\omega t\,dt\right)^2+\left(\int_0^t a(t)\sin\omega t\,dt\right)^2}\right|_{\max} \tag{75}$$

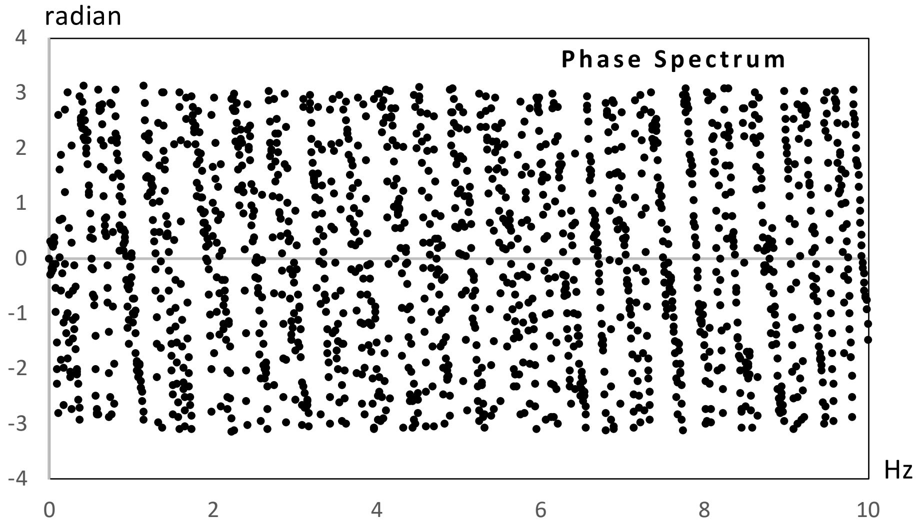
This shows that the following equation holds between the undamped velocity response spectrum $S_V(\omega)_{h=0}$ and the Fourier spectrum of the ground acceleration $\left|F(\omega)\right|$.

$$S_V(\omega)_{h=0} \approx \left|F(\omega)\right| \tag{76}$$

If the time t at which the maximum occurs coincides with the duration T of the seismic motion, the two coincide completely.

The Fourier spectrum and the undamped velocity response spectrum, and phase spectrum for the record of El Centro 1940 (NS component) are shown below.





Reference

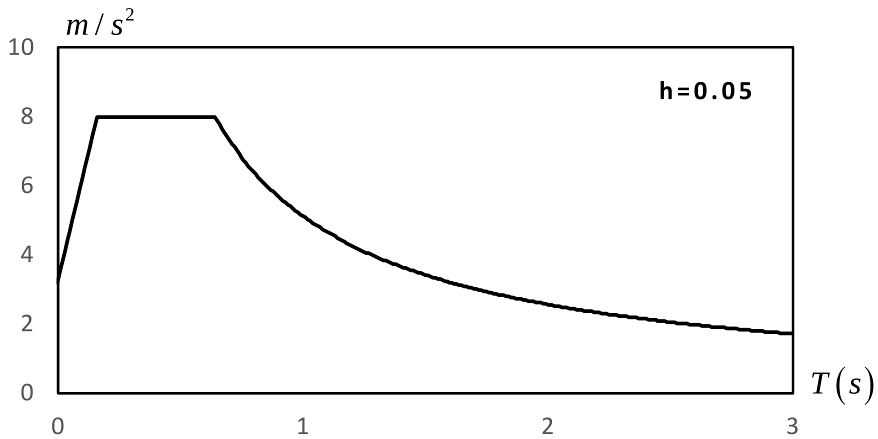
D. E. Hudson, "SOME PROBLEMS IN THE APPLICATION OF SPECTRUM TECHNIQUES TO STRONG-MOTION EARTHQUAKE ANALYSIS", Bulletin of the Seismological Society of America. Vol. 52, No. 2, pp. 417-430. April 1962

6. Generation of artificial earthquake ground motion

1) Target response spectrum

The Building Standard Law of Japan defines the very rare earthquake by the design acceleration response spectrum with 5% damping factor.

$$S_A\left(m/s^2\right)=\begin{cases}3.2+30T & \left(T<0.16s\right)\\ 8.0 & \left(0.16s\le T<0.64s\right)\\ 5.12/T & \left(0.64s\le T\right)\end{cases} \tag{77}$$



On the other hand, for seismic design of high-rise buildings exceeding 60m in height, time history response analysis using ground motion acceleration waveforms is required.

In the following, a method of creating a ground acceleration waveform (artificial ground motion) compatible with the design acceleration response spectrum using the Fourier transform will be described.

2) Fourier spectrum

Given the Fourier spectrum and the phase spectrum,

$$F_k,\phi_k,\quad k=0,1,2,\cdots,N/2 \tag{78}$$

the Fourier coefficient is calculated as

$$C_k=\left|C_k\right|e^{i\phi_k}=\left(F_k/T\right)\left(\cos\phi_k+i\sin\phi_k\right),\quad k=0,1,2,\cdots,N/2 \tag{79}$$

$$C_{N-k}=C_k^{\ *} \tag{80}$$

From the inverse Fourier transform, a time history waveform is obtained as follows.

$$x_m = \sum_{k=0}^{N-1} C_k e^{i\left(\frac{2\pi km}{N}\right)}, \quad m = 0,1,2,\cdots,N-1 \tag{81}$$

For the Fourier spectrum, consider using an undamped velocity response spectrum as a first approximation.

When the acceleration response spectrum with the damping factor $h = 0.05$ is given,

$$S_A\left(f_k\right)_{h=0.05}, \quad k = 0,1,2,\cdots,N/2 \tag{82}$$

The pseudo velocity response spectrum is obtained as

$$S_V\left(f_k\right)_{h=0.05} = S_A\left(f_k\right)_{h=0.05} \Big/ \left(2\pi f_k\right) \tag{83}$$

The undamped velocity response spectrum ($h = 0$) is calculated from

$$S_V\left(f_k\right)_{h=0} = F_h\, S_V\left(f_k\right)_{h=0.05} \tag{84}$$

Where F_h is the modification ratio. The following empirical formula is used.

$$F_h = \frac{1.5}{1+10h} \tag{85}$$

Therefore, the first approximation of the Fourier spectrum is

$$F_k = S_V\left(f_k\right)_{h=0} = F_{h=0}\, S_V\left(f_k\right)_{h=0.05} = 1.5\, S_V\left(f_k\right)_{h=0.05} \tag{86}$$

3) Phase spectrum

There is no specific method to make the phase spectrum, and uniform random numbers or the phase spectrum of an actual earthquake may be used.

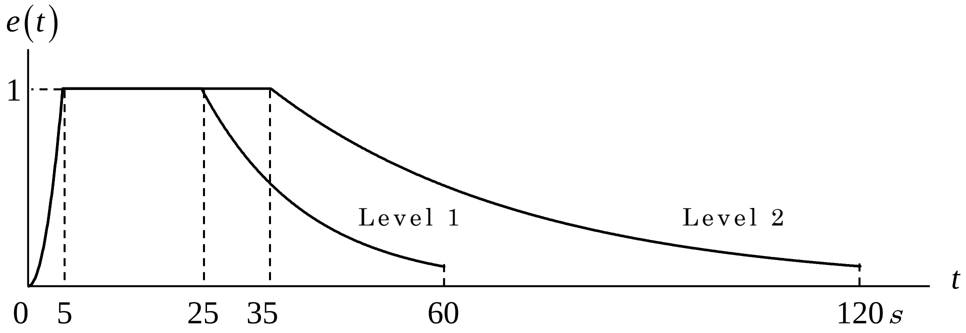
When using uniform random numbers, it is common practice to multiply the time history by an envelope function. The envelope function proposed by Jennings* is adopted in the "Design Technical Guideline for Input Seismic Ground Motion Creation Method" by Building Center of Japan.

Level 1 (Serviceability Limit)

$$e(t) = \begin{cases} (t/5)^2 & (0s < t < 5s) \\ 1.0 & (5s \le t < 25s) \\ \exp\left(-0.066(t-25)\right) & (25s \le t < 60s) \end{cases}$$

Level 2 (Ultimate Limit)

$$e(t) = \begin{cases} (t/5)^2 & (0s < t < 5s) \\ 1.0 & (5s \le t < 35s) \\ \exp\left(-0.027(t-35)\right) & (35s \le t < 120s) \end{cases}$$



*Jennings, P.C., Housner, W.G.. and Tsai, N.C. : Simulated earthquake motions, EERL, Pasadena, 1968

The phase spectrum of an actual earthquake is obtained by the following procedure.

$x_m \left(m = 0, 1, 2, \cdots, N-1\right)$ Read actual earthquake acceleration data

↓ Fourier transform

$C_k = \frac{1}{N}\sum_{m=0}^{N-1} x_m e^{-i\left(\frac{2\pi km}{N}\right)}, \quad k = 0, 1, 2, \cdots, N-1$ Fourier coefficient

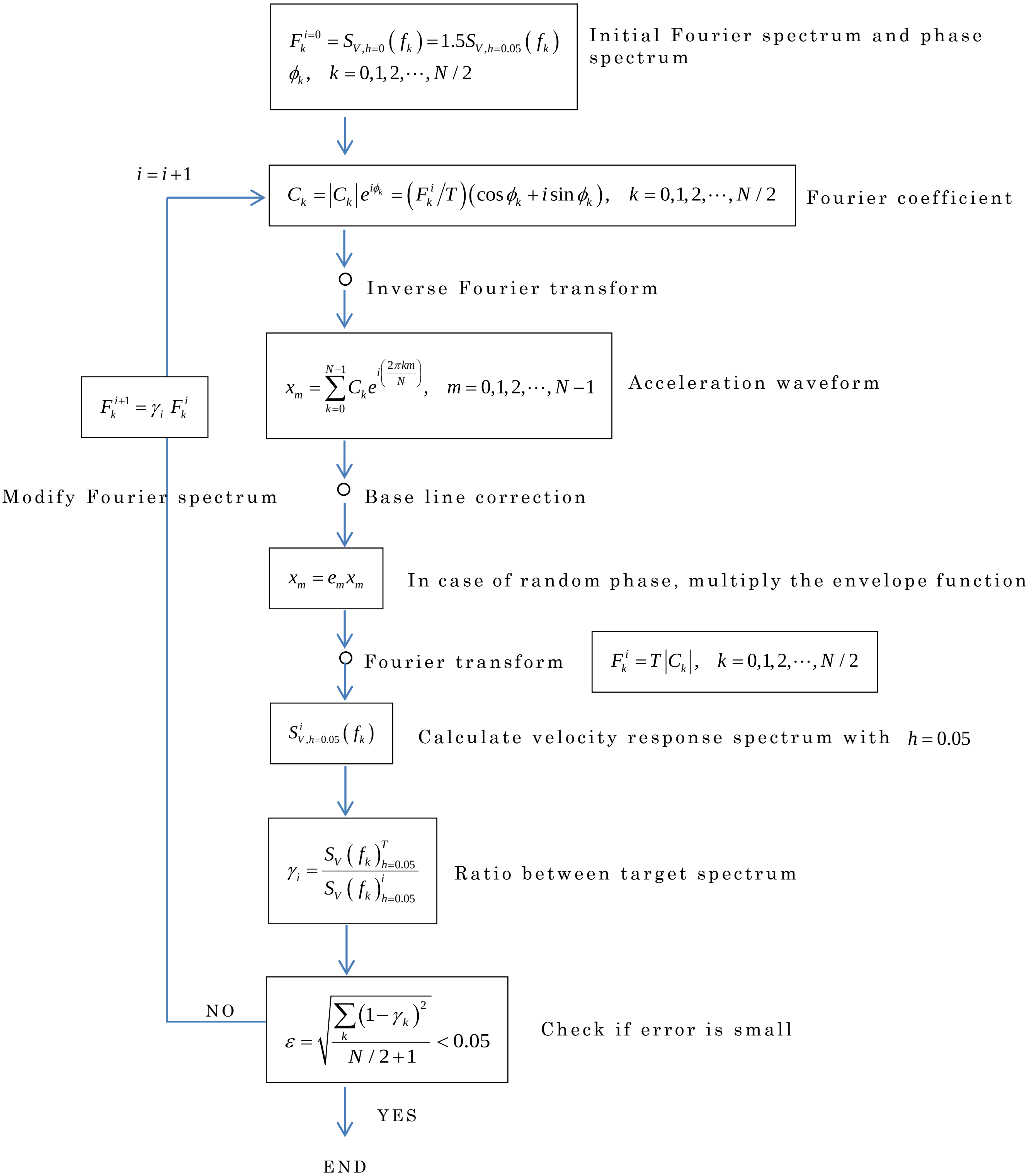
↓

$\phi_k = \tan^{-1}\left(\frac{\mathrm{Re}\left(C_k\right)}{\mathrm{Im}\left(C_k\right)}\right)$ Phase spectrum

Calculate the velocity response spectrum $S_V\left(f_k\right)^i$ of the created simulated ground motion and correct the Fourier spectrum again until the error from the target value $S_V\left(f_k\right)^T$ becomes sufficiently small.

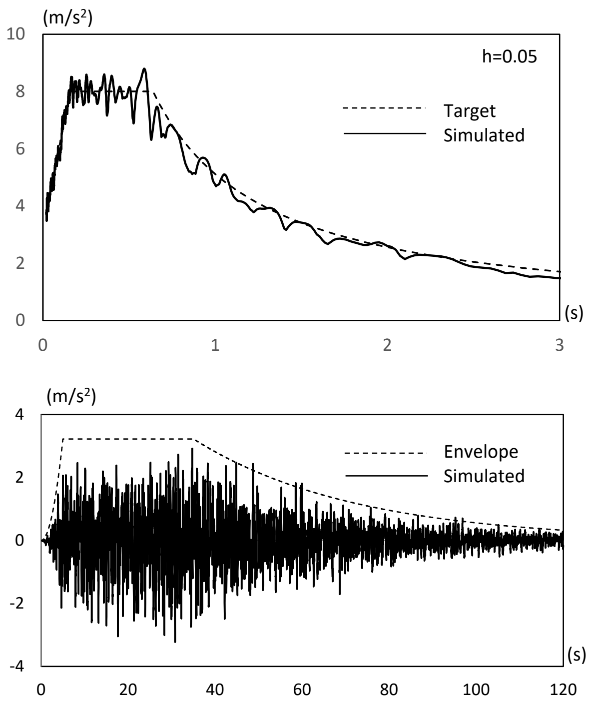
$$F_k^{i+1} = \gamma_i F_k^i, \quad \gamma_i = \frac{S_V\left(f_k\right)_{h=0.05}^T}{S_V\left(f_k\right)_{h=0.05}^i} \tag{87}$$

The flow of creating a simulated ground motion is summarized as follows. The base line correction of the created acceleration waveform is also performed.

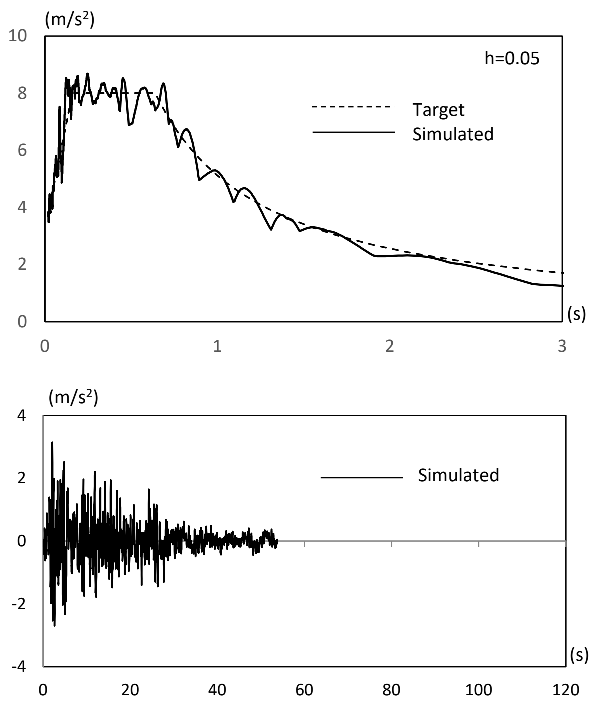


(Example)

The following figures show examples of simulated ground motions those match the design acceleration response spectrum and have different phase spectrum; one is the random phase and another one is the phase of 1995 Kobe earthquake NS component record. In both cases, the acceleration response spectral match the target ones, but the waveform shapes are significantly different.



(a) random phase



(b) phase of 1995 Kobe earthquake

7. Autocorrelation function and Power Spectrum

The average value of the products of functions $x(t)$ and $x(t+\tau)$ is called the **autocorrelation function**, and is expressed as the following equation.

$$R(\tau)=\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt \tag{88}$$

Extending the integration range as

$$R(\tau)=\frac{1}{T}\int_{-\infty}^{\infty} x(t)x(t+\tau)dt \tag{89}$$

The Fourier transform of the autocorrelation function $R(\tau)$ is called the **power spectrum (density function)**.

$$\begin{aligned}
S(\omega)&=\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau \\
&=\int_{-\infty}^{\infty}\left[\frac{1}{T}\int_{-\infty}^{\infty} x(t)x(t+\tau)dt\right]e^{-i\omega\tau}d\tau \\
&=\frac{1}{T}\int_{-\infty}^{\infty} x(t)\left[\int_{-\infty}^{\infty} x(t+\tau)e^{-i\omega(t+\tau)}d\tau\right]e^{i\omega t}dt \\
&=\frac{1}{T}\int_{-\infty}^{\infty} x(t)\left[F(\omega)\right]e^{i\omega t}dt=\frac{1}{T}F(\omega)\int_{-\infty}^{\infty} x(t)e^{i\omega t}dt \\
&=\frac{1}{T}F(\omega)F(-\omega)=\frac{1}{T}\left|F(\omega)\right|^2
\end{aligned} \tag{90}$$

From the above equation, it can be seen that the power spectrum $S(\omega)$ is the average power (square) of the Fourier spectrum $\left|F(\omega)\right|$.

The autocorrelation function is the inverse Fourier transform of the power spectrum.

$$R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{-i\omega\tau}d\omega \tag{91}$$

Note that $S(\omega)=S(-\omega)$ and the same value at positive and negative frequencies, the one-sided power spectrum considering only the positive side frequency is often used.

$$G(\omega)=2S(\omega),\quad \omega>0 \tag{92}$$

8. Generation of a waveform that matches the power spectrum

The flow of generating artificial ground motion in the previous section becomes simpler when creating a waveform that matches the target power spectrum.