STERA_FEM

Technical Manual

Basic Theory of Finite Element Method

Version 2.0

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UPDATE HISTORY

2013/11/11	STERA_FEM Technical Manual Ver.1.1 is uploaded.
2013/12/11	STERA_FEM Technical Manual Ver.1.2 is uploaded.
2014/11/16	STERA_FEM Technical Manual Ver.1.3 is uploaded.
2015/06/11	STERA_FEM Technical Manual Ver.2.0 is uploaded.

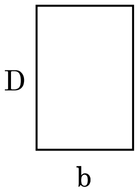
CHAPTER 1

ELASTIC THEORY

Index of Chapter 1

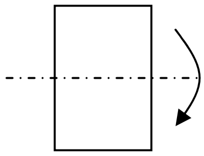
1. INTRODUCTION

1-1. Section

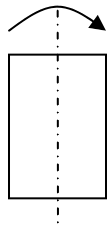


$$A = bD$$

Area



$$I_x = \frac{bD^3}{12}$$

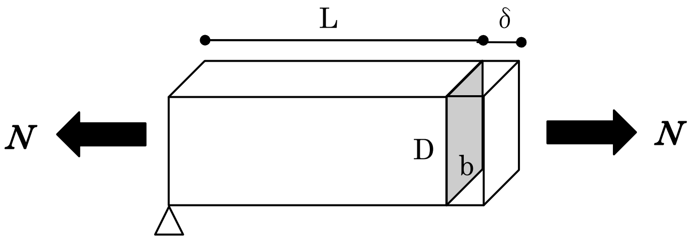


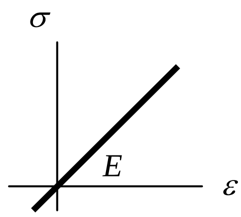
$$I_y = \frac{b^3 D}{12}$$

Moment of Inertia

1-2. Stress and Strain

1) One-Dimensional Problem





$$\sigma = \frac{N}{A}$$

Stress

$$\varepsilon = \frac{\delta}{L}$$

Strain

$$\sigma = E\varepsilon$$

Hooke's Law

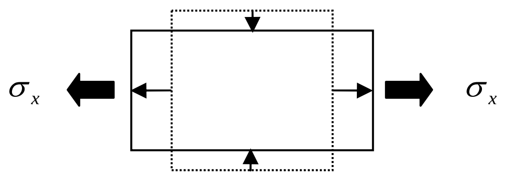
E: Young's Modulus

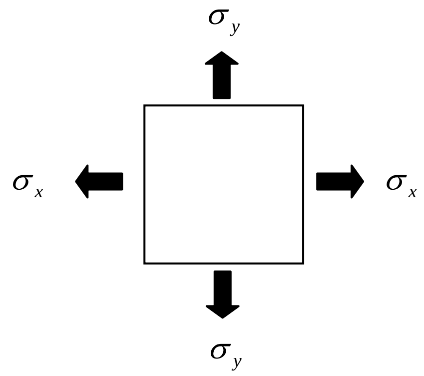
$$N = \frac{EA}{L}\delta$$

Force – Deformation Relationship

2) Two-Dimensional Problem

Normal Stress and Strain





$$\varepsilon_x = \frac{\sigma_x}{E}$$

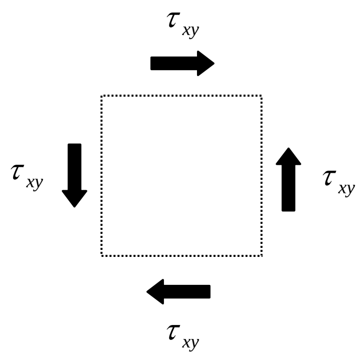
$$\varepsilon_y = -\nu \frac{\sigma_x}{E}$$

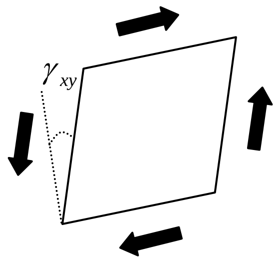
ν : *Poisson Ratio*

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E}$$

Shear Stress and Strain



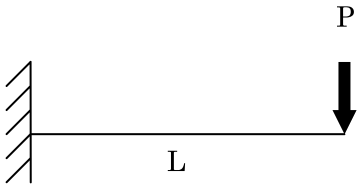


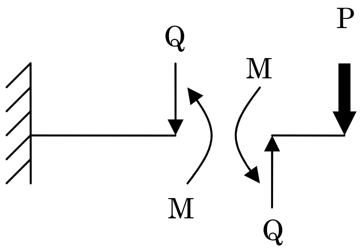
$$\tau_{xy} = G\gamma_{xy}$$

$$G = \frac{1}{2(1+\nu)} E \quad \text{: \textit{Shear Modulus}}$$

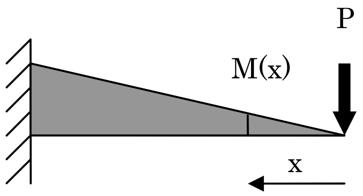
$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{bmatrix} \frac{1}{E} & -\nu \frac{1}{E} & 0 \\ -\nu \frac{1}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

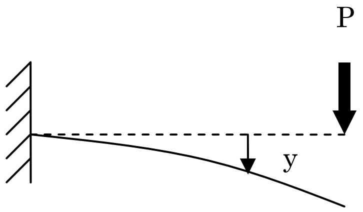
1-3. Beam Theory





Q : Shear Force
M : Moment





$$\frac{d^2y}{dx^2} = \frac{1}{EI}M(x)$$

$$\frac{dM(x)}{dx} = Q(x)$$

Example)

$$\frac{d^2y}{dx^2} = \frac{1}{EI}M(x), \quad M(x) = Px$$

$$\rightarrow \frac{d^2y}{dx^2} = \frac{P}{EI}x$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{2}\frac{P}{EI}x^2 + c_1$$

$$\rightarrow y = \frac{1}{6}\frac{P}{EI}x^3 + c_1x + c_2$$

$at\ x = L,\ \frac{dy}{dx} = 0\ and\ y = 0:$

Therefore,

$$c_1 = -\frac{PL^2}{2EI}, \quad c_2 = \frac{1}{3}\frac{PL^3}{EI}$$

$$y = \frac{1}{6}\frac{P}{EI}x^3 - \frac{1}{2}\frac{PL^2}{EI}x + \frac{1}{3}\frac{PL^3}{EI}$$

1.4 Properties of Reinforced Concrete Structure

Unit Weight

Concrete Type	Nominal Strength (N/mm^2 = MPa)	Unit Weight（kN/m^3）
Normal Concrete	Fc≦36	24

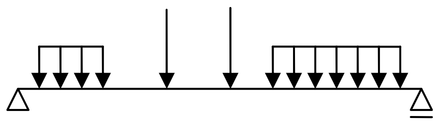
Material Parameters

	Young's Modulus (N/mm^2 = MPa)	Poisson's Ratio	Thermal Expansion Coefficient（1/℃）
Steel Bar	200 000	1/4	1 x 10^{-5}
Concrete	22 000 (Fc = 18) 25 000 (Fc = 24) 28 000 (Fc = 30)	1/6	1 x 10^{-5}

2. SIMPLE EXAMPLE FOR FEM FORMULATION

Step.1: Description of the Problem

The problem is to obtain the deformation of a simple supported beam under various load conditions.



If you change the load condition, you will get the different deformation pattern. Actually, there are infinite variations for the deformation pattern.

etc.

Step.2: Assumption of deformation function

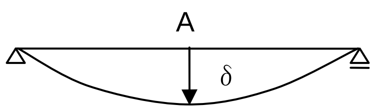
We assume a particular function for the deformation pattern to fix the variation, such as the following function:

$$v(x) = a \sin(\frac{\pi}{L} x) \tag{2-1}$$

Step.3: Relation between nodal displacement and element deformation

From Equation (2-1), The displacement δ at the center node A is calculated as

$$\delta = v(0.5L) = a \tag{2-2}$$



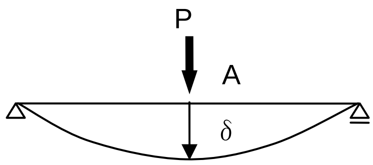
The relation between nodal displacement and element deformation is then expressed as,

$$v(x) = \delta \sin(\frac{\pi}{L} x) \tag{2-3}$$

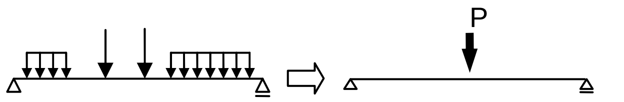
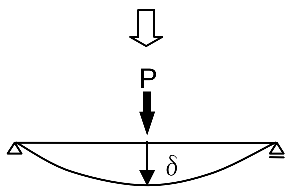
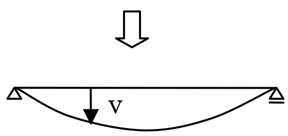
Step.4: Constitutive equation at the node

We obtain the relation between the nodal force and the nodal displacement, for example, by using the “Principle of Virtual Work Method.”

$$P = K\delta \qquad (2\text{-}4)$$



The process is summarized as follows:

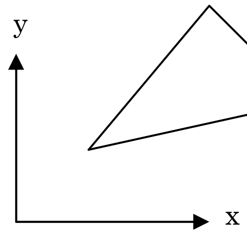
	Translate external forces into equivalent nodal force, P.
	Calculate nodal displacement, δ , from the constitutive equation, $\delta = K^{-1}P$
	Obtain the element deformation from the nodal displacement. $v(x) = \delta \sin(\frac{\pi}{L}x)$

The above example tells the essence of the finite element analysis, which is:
“Assume the deformation pattern to reduce the degree of freedom of the element, then, obtain the deformation from the limited number of nodal displacements.”

3. TRIANGULAR ELEMENT FOR PLANE ANALYSIS

Step.1: Description of the Problem

The problem is to obtain the deformation of a simple triangular element.



There are infinite variations for the deformation patterns.

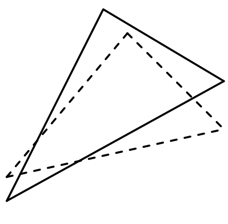
etc.

Step.2: Assumption of deformation function

To fix the variation for the deformation patterns, we assume a linear function for the deformation pattern.

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v(x,y) = \alpha_4 + \alpha_5 x + \alpha_6 y \qquad (3\text{-}1)$$

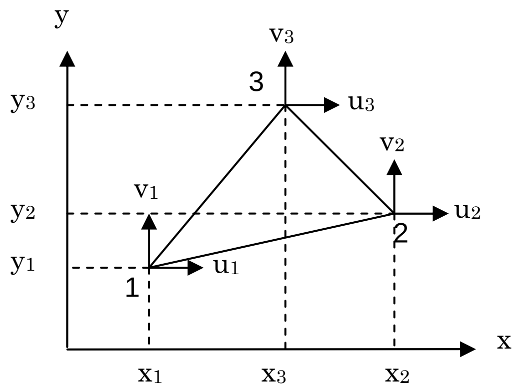
In a matrix form,



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} \qquad (3\text{-}2)$$

Step.3: Relation between nodal displacement and element deformation

The displacements of the element nodes are expressed as,



$$\text{Node 1:}\quad \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} \tag{3-3}$$

$$\text{Node 2:}\quad \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} \tag{3-4}$$

$$\text{Node 3:}\quad \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} \tag{3-5}$$

It is summarized as,

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} \tag{3-6}$$

$$\boldsymbol{U} = \boldsymbol{A} \qquad \boldsymbol{\alpha}$$

We can obtain the coefficients $\alpha_1, \cdots \alpha_6$ from the nodal displacements as,

$$\boldsymbol{\alpha} = \boldsymbol{A}^{-1} \boldsymbol{U} \qquad (3\text{-}7)$$

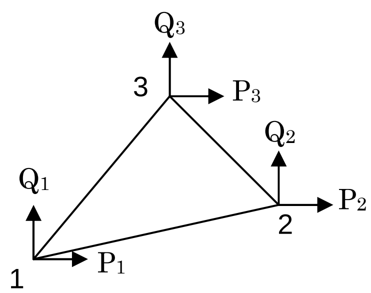
Substituting Equation (3-7) into Equation (3-2), the relation between nodal displacement and element deformation is,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{pmatrix} A^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad (3\text{-}8)$$

$$\boldsymbol{u(x,y)} \quad = \quad \boldsymbol{H(x,y)} \qquad \boldsymbol{U}$$

Step.4: Constitutive equation at the node

We obtain the relation between the nodal force and the nodal displacement, for example, by using the "Principle of Virtual Work Method."



$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = K \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad (3\text{-}9)$$

$$\boldsymbol{F} \quad = \quad \boldsymbol{K}\,\boldsymbol{U}$$

The process is summarized as follows:

(1) Translate external forces into equivalent nodal force,
$\boldsymbol{F} = \{P_1, P_2, P_3, Q_1, Q_2, Q_3\}^T$

(2) Calculate the nodal displacements from the constitutive equation,
$\boldsymbol{U} = \boldsymbol{K}^{-1} \boldsymbol{F}$

(3) Obtain the element deformation from the nodal displacement.
$\boldsymbol{u(x,y)} = \boldsymbol{H(x,y)}\boldsymbol{U}$

4. STIFFNESS MATRIX FOR TRIANGULAR ELEMENT

Stiffness matrix in Equation (3-9) can be obtained from the "*Principle of Virtual Work Method*," which is expressed in the following form:

$$\int_V \bar{\varepsilon}^T \sigma \, dv = \bar{U}^T F \tag{4-1}$$

where, $\bar{\varepsilon}$ is a virtual strain vector, σ is a stress vector, $\bar{U}$ is a virtual displacement vector and F is a load vector, respectively.

In case of the plane problem, the strain ε vector is defined as,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{pmatrix} \tag{4-2}$$

Substituting Equation (3-8) into Equation (4-2), the strain vector is calculated from the nodal displacement vector as,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} A^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{4-3}$$

$$\boldsymbol{\varepsilon} \;=\; \boldsymbol{B} \quad \boldsymbol{U}$$

In the plane stress problem, the stress-strain relationship is expressed as,

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{4-4}$$

$$\boldsymbol{\sigma} \;=\; \boldsymbol{D} \quad \boldsymbol{\varepsilon}$$

Substituting Equation (4-3) into Equation (4-4),

$$\sigma = \boldsymbol{D}\,\boldsymbol{B}\,\boldsymbol{U} \tag{4-5}$$

From the Principle of Virtual Work Method,

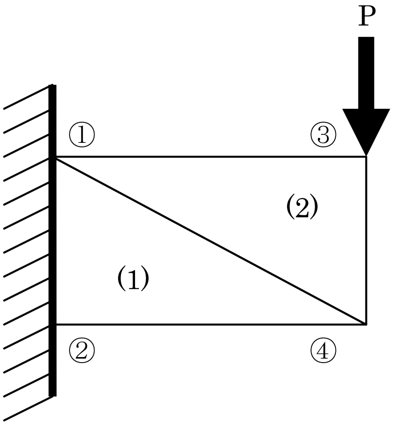
$$\int_V \left(B\overline{U}\right)^T \left(DBU\right) dv = \overline{U}^T \left(\int_V B^T DB dv \right) U = \overline{U}^T F \tag{4-6}$$

Therefore, the constitutive equation is obtained as,

$$F = KU, \quad K = \int_V B^T DB dv \tag{4-7}$$

5. FROM ELEMENT STIFFNESS MATRIX TO GLOBAL STIFFNESS MATRIX

1) Force control



$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \rightarrow \textit{Boundary Condition} \rightarrow \begin{pmatrix} u_3 \\ u_4 \\ v_3 \\ v_4 \end{pmatrix}$$

① fixed … *u1=v1=0*

② fixed … *u2=v2=0*

Element Stiffness Matrix:

Element (1) …

$$\begin{pmatrix} P_1 \\ P_2 \\ P_4 \\ Q_1 \\ Q_2 \\ Q_4 \end{pmatrix} = \begin{pmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} & k_{15}^{(1)} & k_{16}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} & k_{25}^{(1)} & k_{26}^{(1)} \\ k_{31}^{(1)} & k_{32}^{(1)} & k_{33}^{(1)} & k_{34}^{(1)} & k_{35}^{(1)} & k_{36}^{(1)} \\ k_{41}^{(1)} & k_{42}^{(1)} & k_{43}^{(1)} & k_{44}^{(1)} & k_{45}^{(1)} & k_{46}^{(1)} \\ k_{51}^{(1)} & k_{52}^{(1)} & k_{53}^{(1)} & k_{54}^{(1)} & k_{55}^{(1)} & k_{56}^{(1)} \\ k_{61}^{(1)} & k_{62}^{(1)} & k_{63}^{(1)} & k_{64}^{(1)} & k_{65}^{(1)} & k_{66}^{(1)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_4 \\ v_1 \\ v_2 \\ v_4 \end{pmatrix} \quad (5\text{-}1)$$

Element (2) …

$$\begin{pmatrix} P_1 \\ P_3 \\ P_4 \\ Q_1 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} & k_{14}^{(2)} & k_{15}^{(2)} & k_{16}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & k_{23}^{(2)} & k_{24}^{(2)} & k_{25}^{(2)} & k_{26}^{(2)} \\ k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} & k_{34}^{(2)} & k_{35}^{(2)} & k_{36}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)} & k_{44}^{(2)} & k_{45}^{(2)} & k_{46}^{(2)} \\ k_{51}^{(2)} & k_{52}^{(2)} & k_{53}^{(2)} & k_{54}^{(2)} & k_{55}^{(2)} & k_{56}^{(2)} \\ k_{61}^{(2)} & k_{62}^{(2)} & k_{63}^{(2)} & k_{64}^{(2)} & k_{65}^{(2)} & k_{66}^{(2)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \\ u_4 \\ v_1 \\ v_3 \\ v_4 \end{pmatrix} \quad (5\text{-}2)$$

Global Stiffness Matrix:

$$\begin{pmatrix} P_3 \\ P_4 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} k_{22}^{(2)} & k_{23}^{(2)} & k_{25}^{(2)} & k_{26}^{(2)} \\ k_{32}^{(2)} & k_{33}^{(1)} + k_{33}^{(2)} & k_{35}^{(2)} & k_{36}^{(2)} + k_{36}^{(2)} \\ k_{52}^{(2)} & k_{53}^{(2)} & k_{55}^{(2)} & k_{56}^{(2)} \\ k_{62}^{(2)} & k_{63}^{(1)} + k_{63}^{(2)} & k_{65}^{(2)} & k_{66}^{(1)} + k_{66}^{(2)} \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \\ v_3 \\ v_4 \end{pmatrix} \quad (5\text{-}3)$$

F = K U

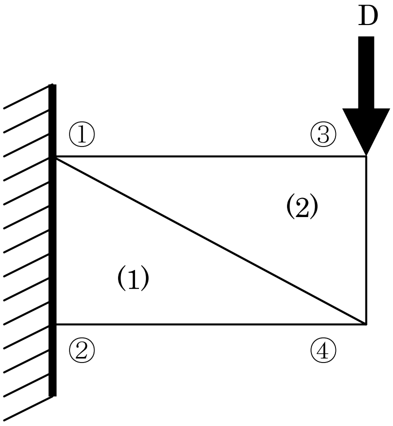
In case of force control, set the load condition,

$$\begin{pmatrix} P_3 \\ P_4 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -P \\ 0 \end{pmatrix} \tag{5-4}$$

The displacement vector is then obtained by solving the constitutive equation,

$$\begin{pmatrix} u_3 \\ u_4 \\ v_3 \\ v_4 \end{pmatrix} = K^{-1} \begin{pmatrix} 0 \\ 0 \\ -P \\ 0 \end{pmatrix} \tag{5-5}$$

2) Displacement Control



$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \rightarrow \textit{Boundary Condition} \rightarrow \begin{pmatrix} u_3 \\ u_4 \\ v_3 \\ v_4 \end{pmatrix}$$

③ fixed … $u_1=v_1=0$

④ fixed … $u_2=v_2=0$

Imposing the control displacement $v_3 = D$ to the Global Stiffness Matrix:

$$\begin{pmatrix} P_3 \\ P_4 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} k_{22}^{(2)} & k_{23}^{(2)} & k_{25}^{(2)} & k_{26}^{(2)} \\ k_{32}^{(2)} & k_{33}^{(1)} + k_{33}^{(2)} & k_{35}^{(2)} & k_{36}^{(2)} + k_{36}^{(2)} \\ k_{52}^{(2)} & k_{53}^{(2)} & k_{55}^{(2)} & k_{56}^{(2)} \\ k_{62}^{(2)} & k_{63}^{(1)} + k_{63}^{(2)} & k_{65}^{(2)} & k_{66}^{(1)} + k_{66}^{(2)} \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \\ D \\ v_4 \end{pmatrix} \tag{5-6}$$

Subtracting the force related to the control displacement from force vector, and other external loads to be zero,

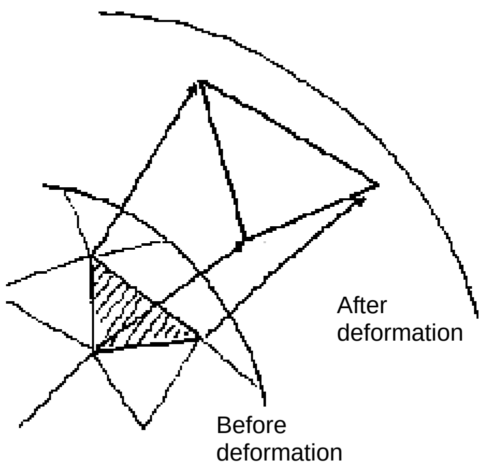
$$\begin{pmatrix} -k_{25}^{(2)} D \\ -k_{35}^{(2)} D \\ -k_{65}^{(2)} D \end{pmatrix} = \begin{pmatrix} k_{22}^{(2)} & k_{23}^{(2)} & k_{26}^{(2)} \\ k_{32}^{(2)} & k_{33}^{(1)} + k_{33}^{(2)} & k_{36}^{(2)} + k_{36}^{(2)} \\ k_{62}^{(2)} & k_{63}^{(1)} + k_{63}^{(2)} & k_{66}^{(1)} + k_{66}^{(2)} \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \\ v_4 \end{pmatrix} \tag{5-7}$$

This process can be done before making the Global Stiffness Matrix. Imposing the control displacement $v_3 = D$ to the element stiffness matrix of Element (2),

$$\begin{pmatrix} P_1 \\ P_3 \\ P_4 \\ Q_1 \\ Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} & k_{14}^{(2)} & k_{15}^{(2)} & k_{16}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & k_{23}^{(2)} & k_{24}^{(2)} & k_{25}^{(2)} & k_{26}^{(2)} \\ k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} & k_{34}^{(2)} & k_{35}^{(2)} & k_{36}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)} & k_{44}^{(2)} & k_{45}^{(2)} & k_{46}^{(2)} \\ k_{51}^{(2)} & k_{52}^{(2)} & k_{53}^{(2)} & k_{54}^{(2)} & k_{55}^{(2)} & k_{56}^{(2)} \\ k_{61}^{(2)} & k_{62}^{(2)} & k_{63}^{(2)} & k_{64}^{(2)} & k_{65}^{(2)} & k_{66}^{(2)} \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \\ u_4 \\ v_1 \\ D \\ v_4 \end{pmatrix} \rightarrow f_{(2)} = \begin{pmatrix} -k_{15}^{(2)} \\ -k_{25}^{(2)} \\ -k_{35}^{(2)} \\ -k_{45}^{(2)} \\ -k_{55}^{(2)} \\ -k_{65}^{(2)} \end{pmatrix} D \tag{5-8}$$

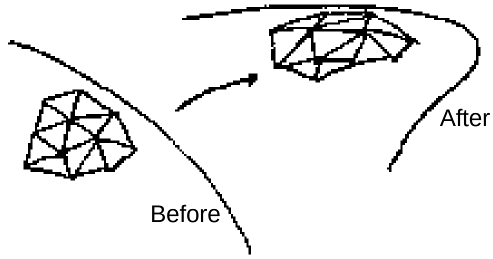
The force vector $f_{(2)}$ will be summed together in the same manner of the element stiffness matrix.

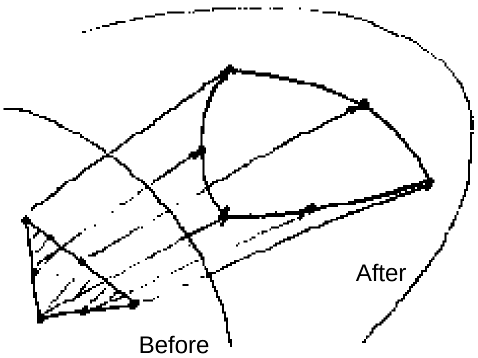
6. HIGHER ORDER ELEMENT



The linear triangular element assumes the deformation pattern to be a linear function between two nodes.

It requires a large number of elements at the place where deformation changes largely.



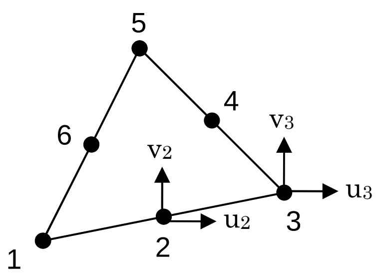


To reduce the number of elements, we introduce the higher order elements, such as the following second order elements where the deformation pattern is assumed to be the second order function of coordinate.

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$
$$v = \alpha_7 + \alpha_8 x + \alpha_9 y + \alpha_{10} x^2 + \alpha_{11} xy + \alpha_{12} y^2 \qquad (6\text{-}1)$$

In a matrix form,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{12} \end{pmatrix} \qquad (6\text{-}2)$$



In order to define the second order function, we need an additional node in the middle of each side of the triangle. At the result, the total number of nodes in one element is 6.

The displacement of the element nodes are then expressed as,

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ - \\ v_1 \\ v_2 \\ \vdots \\ v_6 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & | & & & & & & \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 & | & & & & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & | & & & & & & \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 & | & & & & & & \\ - & - & - & - & - & - & | & - & - & - & - & - & - \\ & & & & & & | & 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ & & & & & & | & 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 \\ & & 0 & & & & | & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & | & 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_6 \\ - \\ \alpha_7 \\ \alpha_8 \\ \vdots \\ \alpha_{12} \end{pmatrix} \qquad (6\text{-}3)$$

$$\boldsymbol{u} \quad = \qquad \boldsymbol{A} \qquad \boldsymbol{\alpha}$$

From Equations (6-1) and (6-2), we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 \end{pmatrix} A^{-1} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ v_2 \\ \vdots \\ v_6 \end{pmatrix} \qquad (6\text{-}4)$$

$$\boldsymbol{u(x,y)} \quad = \qquad \boldsymbol{H(x,y)} \qquad \boldsymbol{U}$$

As the same as the linear triangular element, the constitutive equation is obtained as

$$\begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_6 \\ Q_1 \\ Q_2 \\ \vdots \\ Q_6 \end{pmatrix} = K \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ v_1 \\ v_2 \\ \vdots \\ v_6 \end{pmatrix} \qquad (6\text{-}5)$$

$$\boldsymbol{F} = \boldsymbol{K}\boldsymbol{U}$$

The process is summarized as follows:

(1) Translate external forces into equivalent nodal force,

$\boldsymbol{F} = \{P_1, \ldots, P_6, Q_1, \ldots, Q_6\}^T$

(2) Calculate the nodal displacements from the constitutive equation,

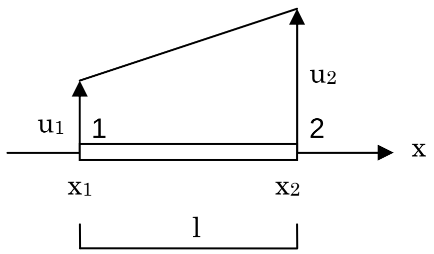
$\boldsymbol{U} = \boldsymbol{K}^{-1} \boldsymbol{F}$

(3) Obtain the element deformation from the nodal displacement.

$\boldsymbol{u(x,y)} = \boldsymbol{H(x,y)U}$

7. INTERPOLATION FUNCTION

Suppose we have one dimensional element under loading. As discussed before, we assume a linear function for the deformation pattern after loading,



$$u(x) = a_0 + a_1 x$$

or

$$u(x) = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \qquad (7\text{-}1)$$

The next step is to obtain the coefficients, a_0, a_1, from the nodal displacements. From the relations:

$$u_1 = a_0 + a_1 x_1$$
$$u_2 = a_0 + a_1 x_2$$

or

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \qquad (7\text{-}2)$$

$$\boldsymbol{U} = \boldsymbol{A}\ \boldsymbol{\alpha}$$

The coefficients are obtained as, $\boldsymbol{\alpha} = \boldsymbol{A}^{-1}\boldsymbol{U}$. Then, the relation between the deformation and the nodal displacements is,

$$u(x) = \begin{pmatrix} 1 & x \end{pmatrix} A^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad (7\text{-}3)$$

Instead of the previous procedure, we introduce the **interpolation functions** to express the deformation directly from the nodal displacements:

$$u(x) = h_1(x)u_1 + h_2(x)u_2 \qquad (7\text{-}4)$$

The interpolation functions, h_1 and h_2, have the following characteristics:

$$h_1(x) = \begin{cases} 1, & x = u_1 \\ 0, & x \neq u_1 \end{cases}, \qquad h_2(x) = \begin{cases} 1, & x = u_2 \\ 0, & x \neq u_2 \end{cases} \qquad (7\text{-}5)$$

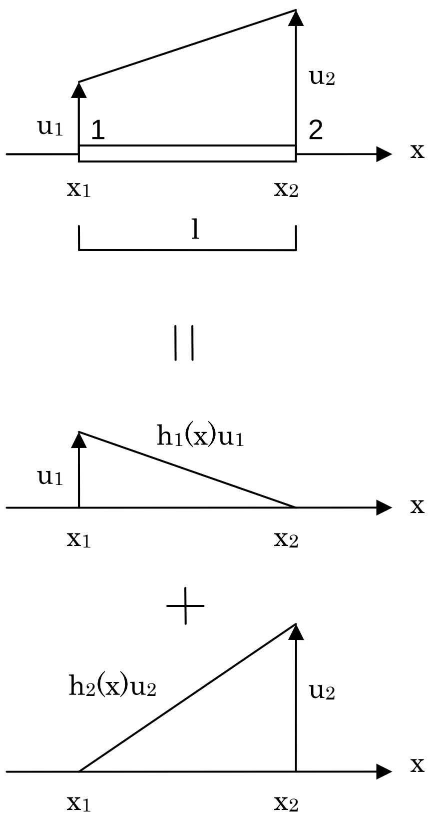
From these characteristics, the functions are easily obtained as,

$$h_1(x) = \frac{x_2 - x}{l}, \qquad h_2(x) = \frac{x - x_1}{l} \tag{7-6}$$

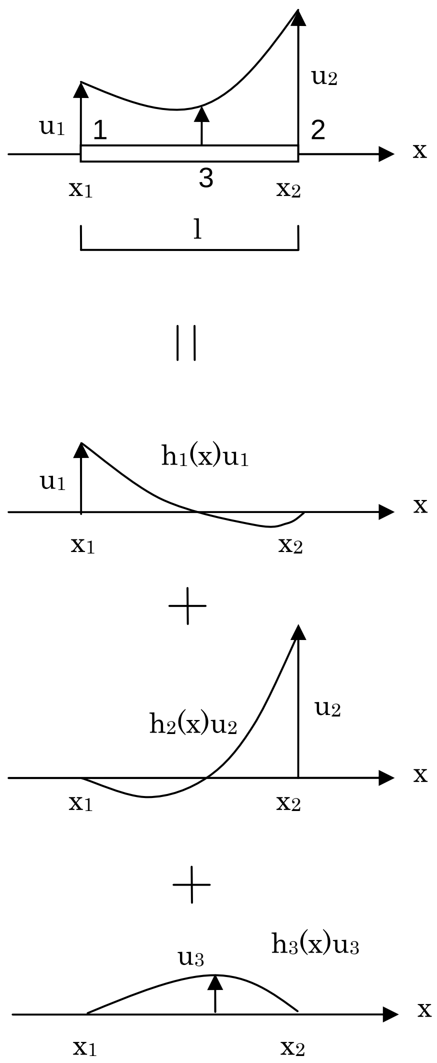
One of the advantages of using interpolation functions is to reduce the burden to calculate the inverse matrix of A in Equation (7-3).

In the same manner, if we assume a second order function for the deformation pattern, the deformation can be directly expressed using interpolation functions as follows:

$$u(x) = h_1(x)u_1 + h_2(x)u_2 + h_3(x)u_3 \tag{7-7}$$



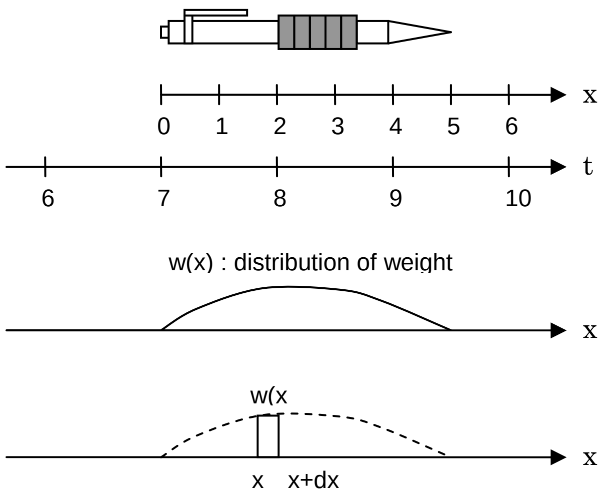
First order interpolation function



Second order interpolation function

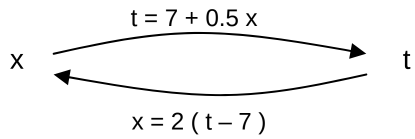
8. NATURAL COORDINATE

1) Natural coordinate



When we measure the coordinate of the pencil, the result is different depending on the scale we use. In this example, the coordinate of the head of the pencil is 5.0 in x-scale and 9.5 in t-scale.

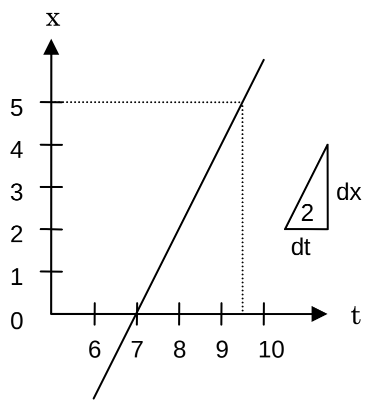
As long as we have one-to-one relationship between two scales, we can translate the value in one scale to the value in another scale anytime.



The total weight of the pencil will be calculated in x-axis as,

$$W = \int_0^5 w(x)dx \qquad (8\text{-}1)$$

To translate it into t-axis, we use the following relationships:



Global relationship:

$$x = 2(t - 7) \qquad (8\text{-}2)$$

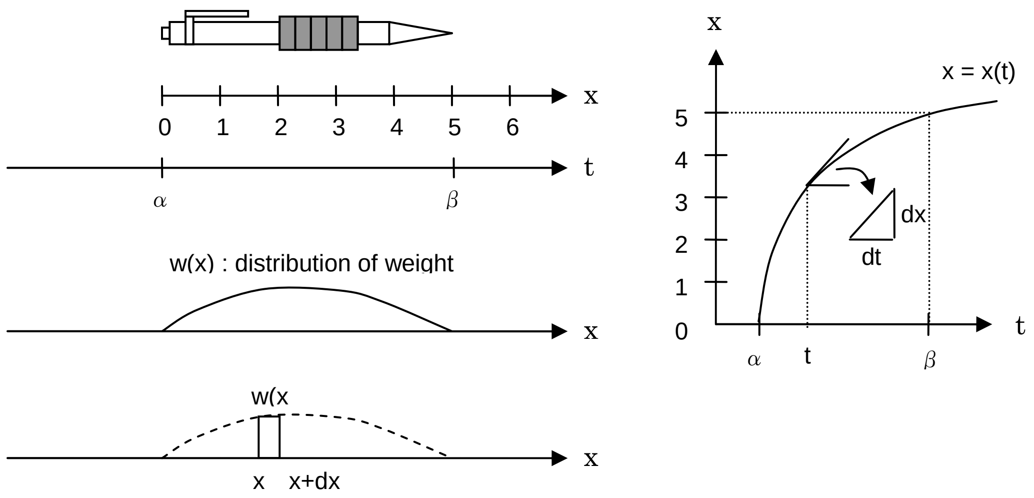
Local relationship:

$$dx = 2dt \qquad (8\text{-}3)$$

Substituting Equations (8-2) and (8-3) into (8-1), the total weight is expressed as,

$$W = 2\int_{7.5}^{9.5} w\big(x(t)\big)dt \qquad (8\text{-}4)$$

Next we consider a more complicated scale to measure the total weight of the pencil.



The relationships between x-axis and t-axis are:

Global relationship: $x = x(t)$ (8-5)

Local relationship: $dx = \frac{dx(t)}{dt}dt$ (8-6)

Where dx(t)/dt represents the first derivative of x(x) by the variable t, which correspond to the slope of x(t) at t. Substituting Equations (8-5) and (8-6) into (8-1), the total weight will be expressed in t-axis as,

$$W = \int_{\alpha}^{\beta} w(x(t))\frac{dx(t)}{dt}dt \qquad (8\text{-}7)$$

Setting α=-1, β=1,

$$W = \int_{-1}^{1} f(t)dt, \quad f(t) = w(x(t))\frac{dx(t)}{dt} \qquad (8\text{-}8)$$

Such coordinate is called "**natural coordinate**."

2) Gaussian quadrature rule

If the integration range is [-1, 1], the integration can be evaluated approximately by n-points Gaussian quadrature rule which is generally expressed in the following form:

$$\int_{-1}^{1} f(t)dt \approx w_1 f(t_1) + w_2 f(t_2) + \cdots + w_n f(t_n) \qquad (8\text{-}9)$$

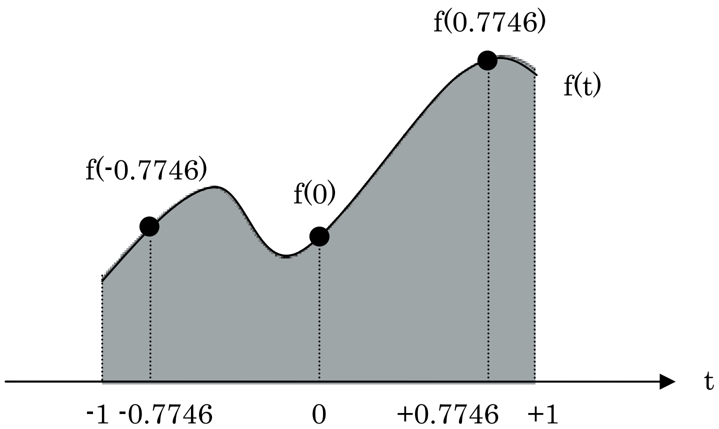
where, $w_1, w_2, \cdots, w_n$ are the weighting coefficients. This formula requires a limited number of function values, $f(t_1), f(t_2), \cdots, f(t_n)$, at the sampling points, $t_1, t_2, \cdots, t_n$, to evaluate the integration.

For example, the **3 points Gaussian quadrature rule** is defined as:

$$\begin{aligned}\int_{-1}^{1} f(t)dt &= 0.5556\ f(-0.7746) + 0.8889\ f(0) + 0.5556\ f(0.7746) \\ &= w_1 f(t_1) + w_2 f(t_2) + w_3 f(t_3)\end{aligned} \qquad (8\text{-}10)$$

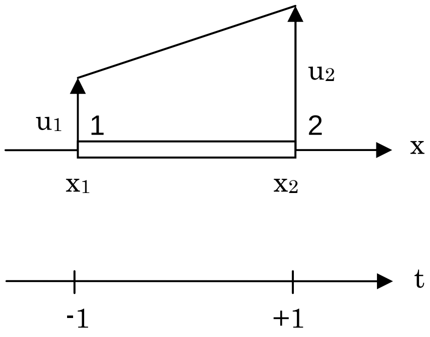
where, $w_1 = 5/9 = 0.5556, \quad w_2 = 8/9 = 0.8889, \quad w_3 = 5/9 = 0.5556$

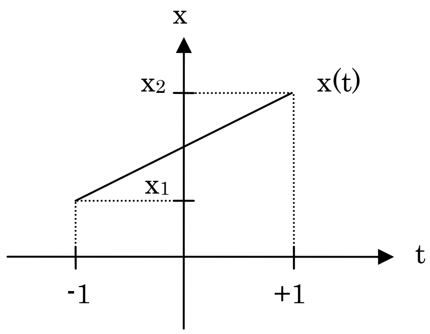
$t_1 = -\sqrt{3/5} = -0.7746, \quad t_2 = 0, \quad t_3 = \sqrt{3/5} = 0.7746$



9. ISOPARAMETRIC ELEMENT

We now introduce the natural coordinate for the example of one dimensional element.





If we assume the linear transfer function x(t) between x-axis and t-axis, x(t) will be expressed as

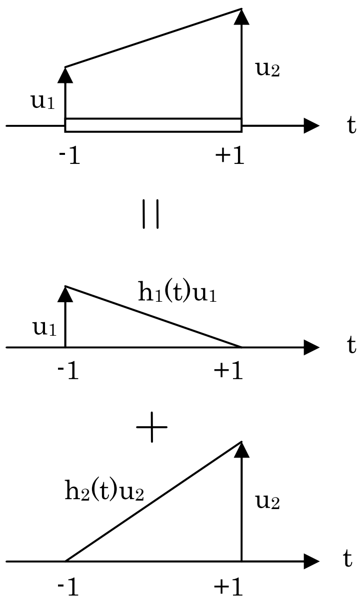
$$x(t) = h_1(t)x_1 + h_2(t)x_2 \tag{9-1}$$

where

$$h_1(t) = \frac{1}{2}(1-t), \quad h_2(t) = \frac{1}{2}(1+t) \tag{9-2}$$

Actually, it satisfies the fact that

$$x(-1) = x_1, \quad x(1) = x_2 \tag{9-3}$$



The deformation of the element is also expressed as,

$$u(t) = h_1(t)u_1 + h_2(t)u_2 \tag{9-4}$$

Therefore, the functions $h_1(t), h_2(t)$ are the interpolation functions we introduced before.

The element where both the coordinate transfer function x(t) and the deformation function u(t) are expressed using the same interpolation functions on the natural coordinate is called "**Isoparametric element.**"

Advantages of using isplarametric elements are summarized below:

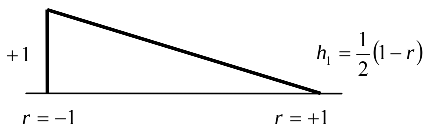
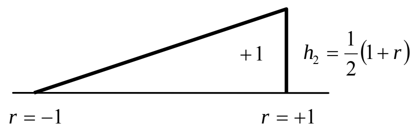
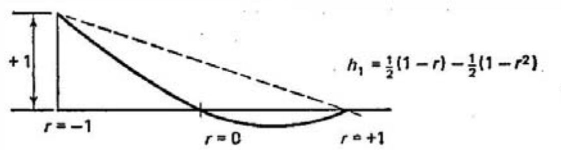
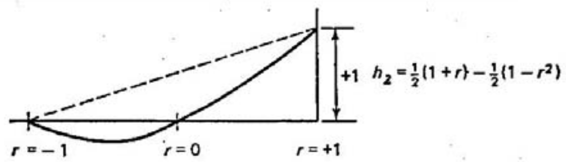
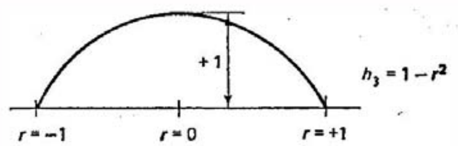
(1) The relation $u(t) = \sum_{i=1}^{n} h_i(t) u_i$ does not require the calculation of inverse matrix.

(2) The relation $x(t) = \sum_{i=1}^{n} h_i(t) x_i$ enables to use the numerical integration method.

(3) Both functions u(t) and x(t) are expressed using the same interpolation functions.

10. SYSTEMATIC FORMULATION OF INTERPOLATION FUNCTION

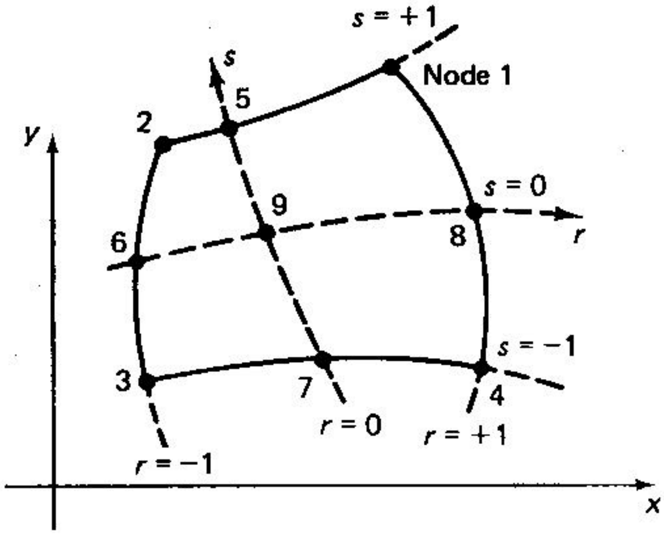
(1) One dimensional element

<table>
<tr>
<td rowspan="2">2 Node</td>
<td>
</td>
<td>$h_1 = \frac{1}{2}(1-r)$</td>
<td rowspan="2"></td>
</tr>
<tr>
<td>
</td>
<td>$h_2 = \frac{1}{2}(1+r)$</td>
</tr>
<tr>
<td rowspan="3">3 Node</td>
<td>
</td>
<td>$h_1 = \frac{1}{2}(1-r)$</td>
<td>$-\frac{1}{2}(1-r^2)$</td>
</tr>
<tr>
<td>
</td>
<td>$h_2 = \frac{1}{2}(1+r)$</td>
<td>$-\frac{1}{2}(1-r^2)$</td>
</tr>
<tr>
<td>
</td>
<td>$h_3 = 1-r^2$</td>
<td></td>
</tr>
</table>

As presented here, if you increase a node to define the second order function for the deformation, the interpolation function changes in the following manners:

- Modify the existing interpolation functions, h_1 and h_2,
- Define a new interpolation function, h_3.

(2) Two dimensional element



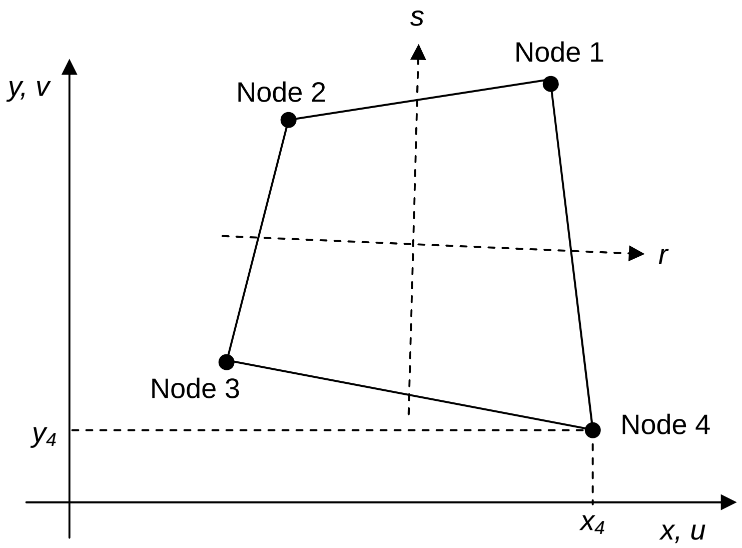
(a) Four to 9 variable-number-nodes two-dimensional element

		Include only if node i is defined				
		$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$
$h_1 =$	$\frac{1}{4}(1+r)(1+s)$	$-\frac{1}{2}h_5$	....	....	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_2 =$	$\frac{1}{4}(1-r)(1+s)$	$-\frac{1}{2}h_5$	$-\frac{1}{2}h_6$	....	....	$-\frac{1}{4}h_9$
$h_3 =$	$\frac{1}{4}(1-r)(1-s)$	....	$-\frac{1}{2}h_6$	$-\frac{1}{2}h_7$	....	$-\frac{1}{4}h_9$
$h_4 =$	$\frac{1}{4}(1+r)(1-s)$	....	....	$-\frac{1}{2}h_7$	$-\frac{1}{2}h_8$	$-\frac{1}{4}h_9$
$h_5 =$	$\frac{1}{2}(1-r^2)(1+s)$	....	....	....	....	$-\frac{1}{2}h_9$
$h_6 =$	$\frac{1}{2}(1-s^2)(1-r)$	....	....	....	....	$-\frac{1}{2}h_9$
$h_7 =$	$\frac{1}{2}(1-r^2)(1-s)$	....	....	....	....	$-\frac{1}{2}h_9$
$h_8 =$	$\frac{1}{2}(1-s^2)(1+r)$	....	....	....	....	$-\frac{1}{2}h_9$
$h_9 =$	$(1-r^2)(1-s^2)$					

(b) Interpolation functions

11. STIFFNESS MATRIX FOR ISOPARAMETRIC ELEMENT

A two dimensional isoparametric element is adopted below.



The coordinate transfer function {x, y} is expressed using the interpolation functions as

$$x(r,s)=\sum_{i=1}^{4}h_i(r,s)x_i=\frac{1}{4}(1+r)(1+s)x_1+\frac{1}{4}(1-r)(1+s)x_2+\frac{1}{4}(1-r)(1-s)x_3+\frac{1}{4}(1+r)(1-s)x_4$$

$$y(r,s)=\sum_{i=1}^{4}h_i(r,s)y_i=\frac{1}{4}(1+r)(1+s)y_1+\frac{1}{4}(1-r)(1+s)y_2+\frac{1}{4}(1-r)(1-s)y_3+\frac{1}{4}(1+r)(1-s)y_4$$

(11-1)

The deformation function {u, v} is also expressed using the same interpolation functions.

$$u(r,s)=\sum_{i=1}^{4}h_i(r,s)u_i=\frac{1}{4}(1+r)(1+s)u_1+\frac{1}{4}(1-r)(1+s)u_2+\frac{1}{4}(1-r)(1-s)u_3+\frac{1}{4}(1+r)(1-s)u_4$$

$$v(r,s)=\sum_{i=1}^{4}h_i(r,s)v_i=\frac{1}{4}(1+r)(1+s)v_1+\frac{1}{4}(1-r)(1+s)v_2+\frac{1}{4}(1-r)(1-s)v_3+\frac{1}{4}(1+r)(1-s)v_4$$

(11-2)

In a matrix form,

$$\begin{pmatrix}u(r,s)\\v(r,s)\end{pmatrix}=\begin{bmatrix}h_1(r,s) & 0 & h_2(r,s) & 0 & h_3(r,s) & 0 & h_4(r,s) & 0\\ 0 & h_1(r,s) & 0 & h_2(r,s) & 0 & h_3(r,s) & 0 & h_4(r,s)\end{bmatrix}\begin{pmatrix}u_1\\v_1\\u_2\\v_2\\u_3\\v_3\\u_4\\v_4\end{pmatrix}$$

$$\boldsymbol{u(r,s)=H(r,s)\,U}\qquad(11\text{-}3)$$

Stiffness matrix can be obtained from the "*Principle of Virtual Work Method*," which is expressed in the following form:

$$\int_V \bar{\varepsilon}^T \sigma \, dv = \overline{U}^T F \tag{11-4}$$

where, $\bar{\varepsilon}$ is a virtual strain vector, σ is a stress vector, $\overline{U}$ is a virtual displacement vector and F is a load vector, respectively. In case of the plane problem, the strain ε vector is defined as,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{pmatrix} \tag{11-5}$$

Substituting Equation (11-2), the strain vector is calculated as,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial x} u_i \\ \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial y} v_i \\ \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial y} u_i + \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial x} v_i \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\partial h_1}{\partial x} & 0 & \dfrac{\partial h_2}{\partial x} & 0 & \dfrac{\partial h_3}{\partial x} & 0 & \dfrac{\partial h_4}{\partial x} & 0 \\ 0 & \dfrac{\partial h_1}{\partial y} & 0 & \dfrac{\partial h_2}{\partial y} & 0 & \dfrac{\partial h_3}{\partial y} & 0 & \dfrac{\partial h_4}{\partial y} \\ \dfrac{\partial h_1}{\partial y} & \dfrac{\partial h_1}{\partial x} & \dfrac{\partial h_2}{\partial y} & \dfrac{\partial h_2}{\partial x} & \dfrac{\partial h_3}{\partial y} & \dfrac{\partial h_3}{\partial x} & \dfrac{\partial h_4}{\partial y} & \dfrac{\partial h_4}{\partial x} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}$$

$$\varepsilon \quad = \quad \boldsymbol{B} \quad \boldsymbol{U} \tag{11-6}$$

In the plane stress problem, the stress-strain relationship is expressed as,

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{11-7}$$

$$\sigma \quad = \quad \boldsymbol{D} \quad \varepsilon$$

Substituting Equation (11-6) into Equation (11-7),

$$\sigma = D\,B\,U \tag{11-8}$$

From the Principle of Virtual Work Method,

$$\int_V \left(B\overline{U}\right)^T (DBU)dv = \overline{U}^T \left(\int_V B^T DBdv \right) U = \overline{U}^T F \tag{11-9}$$

Therefore, the constitutive equation is obtained as,

$$F = KU, \quad K = \int_V B^T DBdv \tag{11-10}$$

If we assume the constant thickness of the plate (= t), using the relation $dv = tdxdy$,

$$K = t \int_{V(x,y)} B^T DBdxdy \tag{11-11}$$

The following relationship is also derived from Equation (11-9)

$$\int_V \left(B\overline{U}\right)^T \{\sigma\} dv = \overline{U}^T \left(\int_V B^T \{\sigma\} dv \right) = \overline{U}^T F \tag{11-12}$$

Therefore, the relationship between stress vector and nodal force is

$$F = \int_V B^T \{\sigma\} dv \tag{11-13}$$

Since this integration is defined in x-y coordinate, we must transfer the coordinate into r-s coordinate to use the numerical integration method. Introducing the **Jacobian matrix**,

$$J = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{pmatrix} ; \textit{Jacobian Matrix} \tag{11-14}$$

the above integration is expressed in r-s coordinate as,

$$K = t \int_{-1}^{1} \int_{-1}^{1} B(x(r,s), y(r,s))^T DB(x(r,s), y(r,s)) \frac{\partial(x,y)}{\partial(r,s)} drds \tag{11-15}$$

where

$$\frac{\partial(x,y)}{\partial(r,s)} = \det J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{vmatrix} \tag{11-16}$$

1) Evaluation of Jacobian Matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{4} \frac{\partial h_i}{\partial r} x_i & \sum_{i=1}^{4} \frac{\partial h_i}{\partial r} y_i \\ \sum_{i=1}^{4} \frac{\partial h_i}{\partial s} x_i & \sum_{i=1}^{4} \frac{\partial h_i}{\partial s} y_i \end{pmatrix} \tag{11-17}$$

2) Evaluation of the matrix B

From Equation (11-6),

$$B = \begin{pmatrix} \frac{\partial h_1}{\partial x} & 0 & \frac{\partial h_2}{\partial x} & 0 & \frac{\partial h_3}{\partial x} & 0 & \frac{\partial h_4}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \frac{\partial h_2}{\partial y} & 0 & \frac{\partial h_3}{\partial y} & 0 & \frac{\partial h_4}{\partial y} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial y} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial y} & \frac{\partial h_4}{\partial x} \end{pmatrix} \tag{11-18}$$

The derivatives $\frac{\partial h_1}{\partial x}, \cdots, \frac{\partial h_4}{\partial x}, \frac{\partial h_1}{\partial y}, \cdots, \frac{\partial h_4}{\partial y}$ are calculated as,

$$\frac{\partial h_1}{\partial x} = \frac{\partial h_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial h_1}{\partial s}\frac{\partial s}{\partial x}, \quad \cdots \quad , \frac{\partial h_4}{\partial x} = \frac{\partial h_4}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial h_4}{\partial s}\frac{\partial s}{\partial x},$$
$$\frac{\partial h_1}{\partial y} = \frac{\partial h_1}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial h_1}{\partial s}\frac{\partial s}{\partial y}, \quad \cdots \quad , \frac{\partial h_4}{\partial y} = \frac{\partial h_4}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial h_4}{\partial s}\frac{\partial s}{\partial y}$$

In a matrix form,

$$\begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial x} & \frac{\partial h_3}{\partial x} & \frac{\partial h_4}{\partial x} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_2}{\partial y} & \frac{\partial h_3}{\partial y} & \frac{\partial h_4}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial h_1}{\partial r} & \frac{\partial h_2}{\partial r} & \frac{\partial h_3}{\partial r} & \frac{\partial h_4}{\partial r} \\ \frac{\partial h_1}{\partial s} & \frac{\partial h_2}{\partial s} & \frac{\partial h_3}{\partial s} & \frac{\partial h_4}{\partial s} \end{pmatrix}$$

$$= J^{-1} \begin{pmatrix} \frac{\partial h_1}{\partial r} & \frac{\partial h_2}{\partial r} & \frac{\partial h_3}{\partial r} & \frac{\partial h_4}{\partial r} \\ \frac{\partial h_1}{\partial s} & \frac{\partial h_2}{\partial s} & \frac{\partial h_3}{\partial s} & \frac{\partial h_4}{\partial s} \end{pmatrix} \tag{11-19}$$

3) Evaluation of partial derivatives of the interpolation functions

$$\begin{aligned}
\frac{\partial h_1}{\partial r} &= \frac{1}{4}(1+s) & \frac{\partial h_1}{\partial s} &= \frac{1}{4}(1+r) \\
\frac{\partial h_2}{\partial r} &= -\frac{1}{4}(1+s) & \frac{\partial h_2}{\partial s} &= \frac{1}{4}(1-r) \\
\frac{\partial h_3}{\partial r} &= -\frac{1}{4}(1-s) , & \frac{\partial h_3}{\partial s} &= -\frac{1}{4}(1-r) \\
\frac{\partial h_4}{\partial r} &= \frac{1}{4}(1-s) & \frac{\partial h_4}{\partial s} &= -\frac{1}{4}(1+s)
\end{aligned} \qquad (11\text{-}20)$$

4) Numerical integration

Using the 3 points Gaussian quadrature rule, the stiffness matrix is calculated numerically as follows:

$$\begin{aligned}
K &= t\int_{-1}^{1}\int_{-1}^{1} B(x(r,s),y(r,s))^T DB(x(r,s),y(r,s))\frac{\partial(x,y)}{\partial(r,s)}drds \\
&= t\int_{-1}^{1}\int_{-1}^{1} F(r,s)drds \\
&= t\sum_{i=1}^{3}\sum_{j=1}^{3} w_i w_j F(r_i,s_j)
\end{aligned} \qquad (11\text{-}21)$$

where

$$F(r,s) = B(x(r,s),y(r,s))^T DB(x(r,s),y(r,s))\frac{\partial(x,y)}{\partial(r,s)}$$

$$w_1 = 5/9 = 0.5556, \quad w_2 = 8/9 = 0.8889, \quad w_3 = 5/9 = 0.5556$$

$$r_1 = s_1 = -\sqrt{3/5} = -0.7746, \quad r_2 = s_2 = 0, \quad r_3 = s_3 = \sqrt{3/5} = 0.7746$$

5) Assemble of finite element

To total stiffness matrix can be obtained to assemble the element stiffness matrix over the areas of all finite elements.

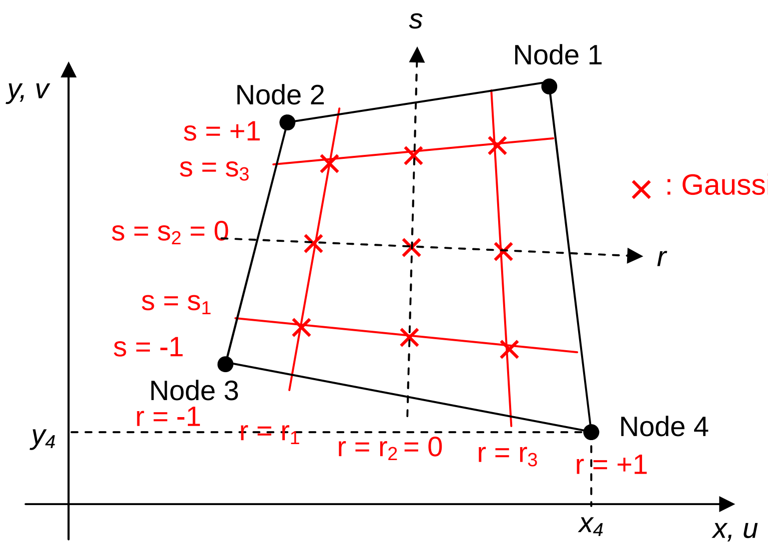
$$K = \sum_m K_m \qquad (11\text{-}22)$$

where m denotes the m-th element.

12. STRESS AND STRAIN AT GAUSSIAN POINTS

1) Stress and strain at Gaussian point

If you use the 3-points Gaussian Integration Method, there are nine Gaussian points $(r_i, s_j)\,(i = 1,2,3,\ j = 1,2,3)$ in an element.



The stress and strain at the Gaussian point, (r_i, s_j), is obtained from Equations (11-5) and (11-7) as

$$\sigma_{ij} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}_{ij} = DB_{ij}U \tag{12-1}$$

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}_{ij} = B_{ij}U \tag{12-2}$$

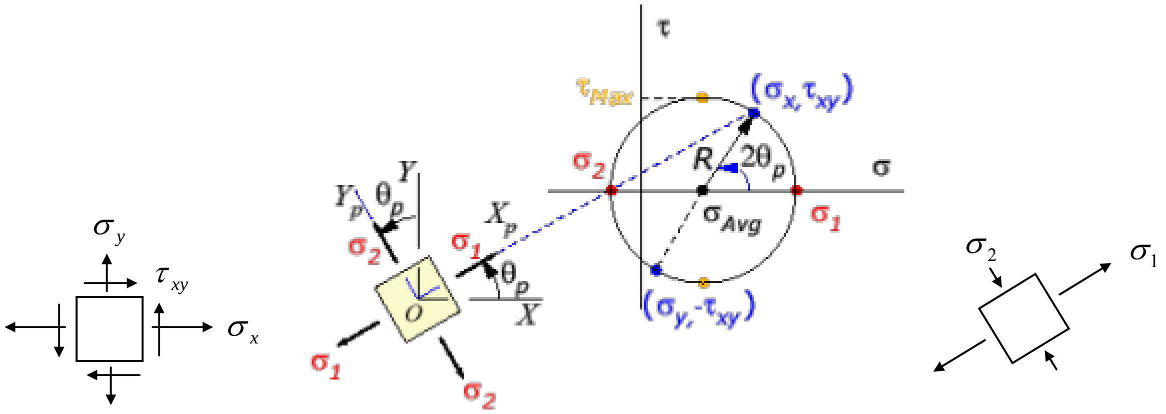
2) Principal stress at Gaussian point

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + {\tau_{xy}}^2} \qquad (12\text{-}3)$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + {\tau_{xy}}^2} \qquad (12\text{-}4)$$

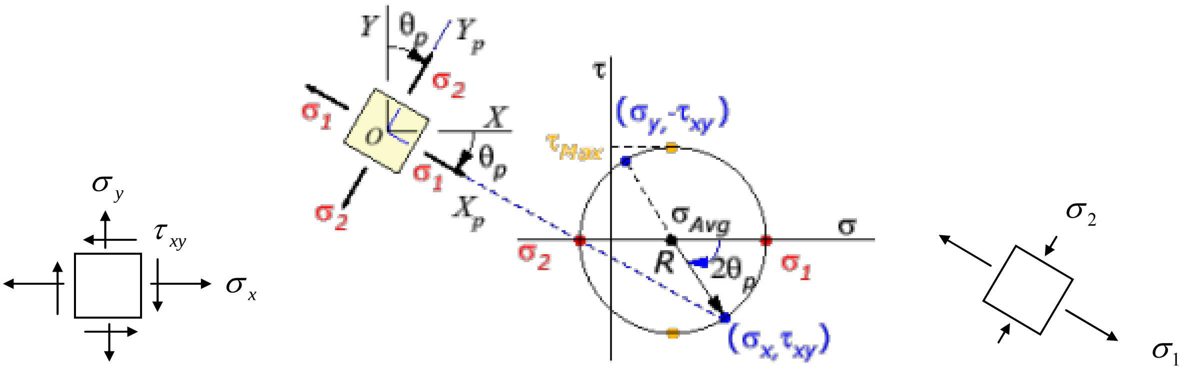
$$\theta_p = \frac{1}{2}\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) \qquad (12\text{-}5)$$

Case 1: $\tau_{xy} > 0$ and $\sigma_x > \sigma_y$



Note) Range of arctan is [-π/2, π/2], and 2θp is in this range.

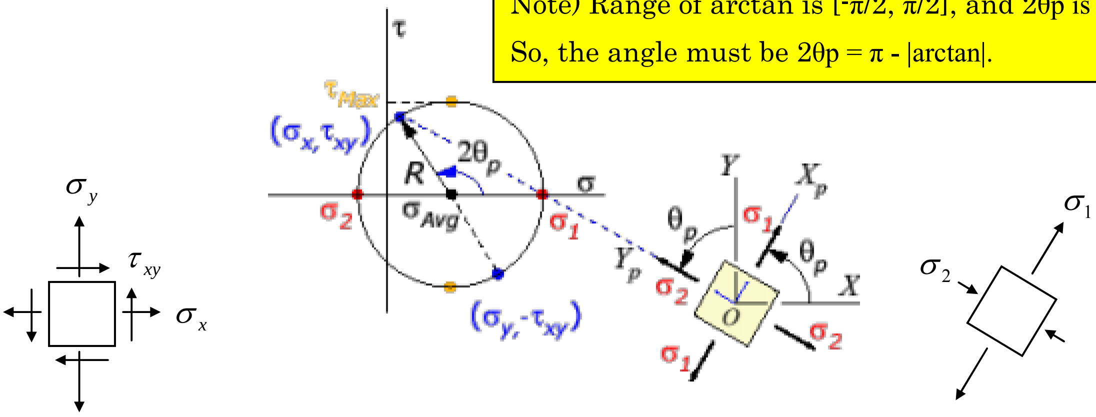
Case 2: $\tau_{xy} < 0$ and $\sigma_x > \sigma_y$



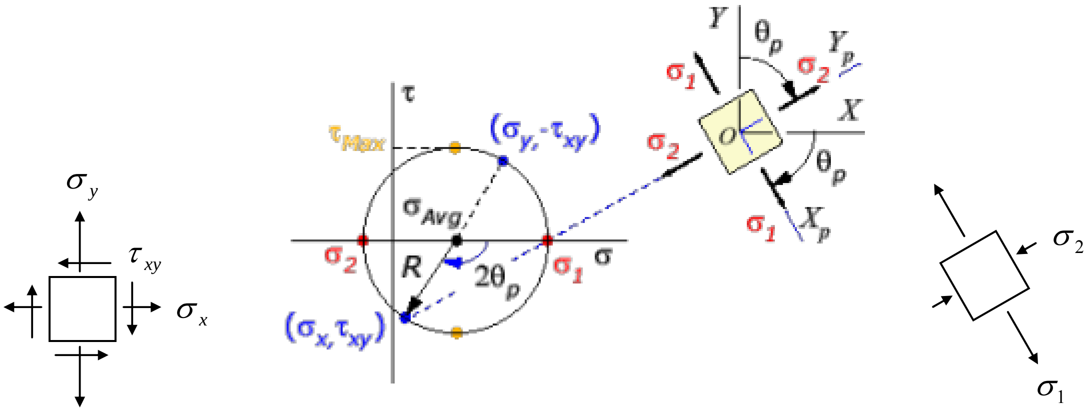
Note) Range of arctan is [-π/2, π/2], and 2θp is in this range.

Case 3: $\tau_{xy} > 0$ and $\sigma_x < \sigma_y$

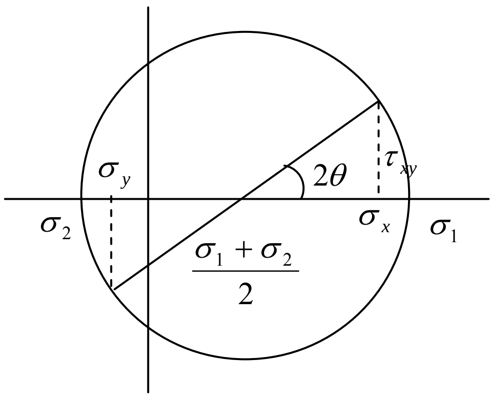
Note) Range of arctan is [-π/2, π/2], and 2θp is not in this range.
So, the angle must be 2θp = π - |arctan|.



Case 4: $\tau_{xy} < 0$ and $\sigma_x < \sigma_y$



Note) Range of arctan is [-π/2, π/2], and 2θp is not in this range.
So, the angle must be 2θp = arctan - π.



$$\sigma_x = \frac{\sigma_1 + \sigma_2}{2} + \left(\frac{\sigma_1 - \sigma_2}{2}\right)\cos(2\theta)$$

$$\sigma_y = \frac{\sigma_1 + \sigma_2}{2} - \left(\frac{\sigma_1 - \sigma_2}{2}\right)\cos(2\theta)$$

$$\tau_{xy} = \left(\frac{\sigma_1 - \sigma_2}{2}\right)\sin(2\theta) \tag{12-6}$$

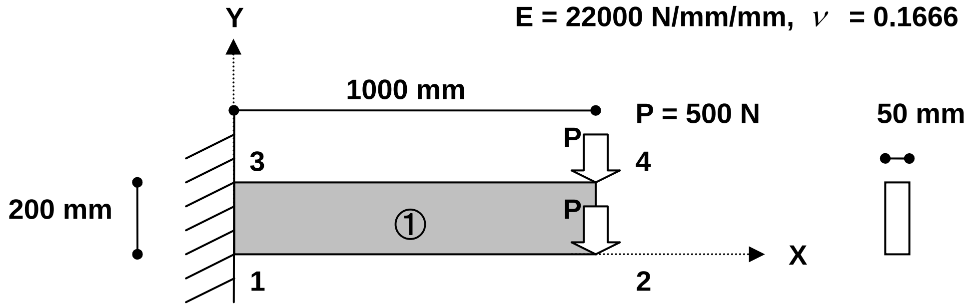
3) Displacement at Gaussian point

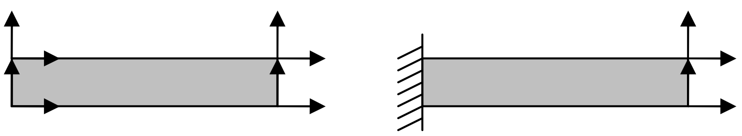
After obtaining the nodal displacement, the displacement at the Gaussian point $\left(r_i, s_j\right)$, is obtained from Equation (11-2) as

$$
\begin{aligned}
u(r_i, s_j) &= \sum_{i=1}^{4} h_i(r_i, s_j) u_i \\
v(r_i, s_j) &= \sum_{i=1}^{4} h_i(r_i, s_j) v_i
\end{aligned}
\tag{12-7}
$$

13. INDEPENDENT FREEDOM

1) Freedom Vector



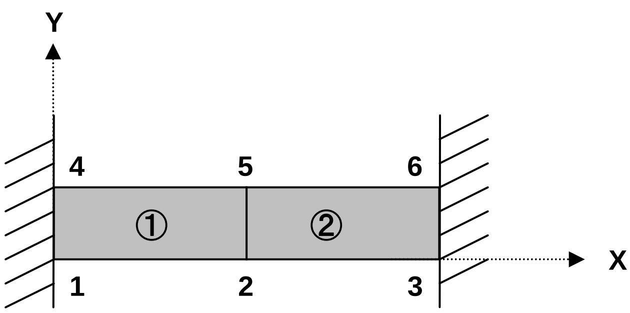


$\{F\} =$ (Initial → Restrained freedom = 1 → Numbering)

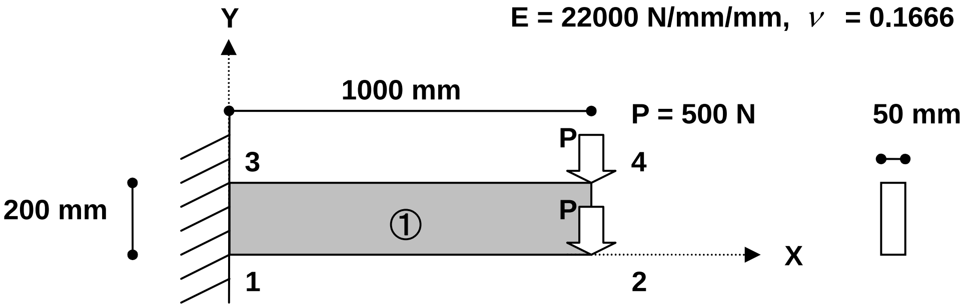
	Initial	Restrained freedom = 1	Numbering
1_X	0	1	0
1_Y	0	1	0
2_X	0	0	1
2_Y	0	0	2
3_X	0	1	0
3_Y	0	1	0
4_X	0	0	3
4_Y	0	0	4

Exercise 1)

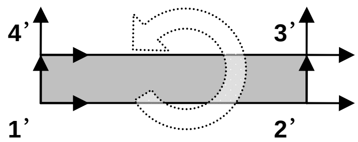
Please obtain the freedom vector of the following structure.



2) Location Matrix



①Element stiffness matrix



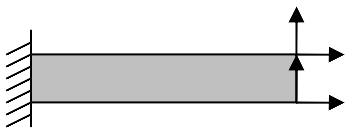
Element node number should be in anti-clockwise order.

		1’X	1’Y	2’X	2’Y	3’X	3’Y	4’X	4’Y	
P1’		K11	K12	K13	K14	K15	K16	K17	K18	U1’
Q1’		K21	K22	K23	K24	K25	K26	K27	K28	V1’
P2’		K31	K32	K33	K34	K35	K36	K37	K38	U2’
Q2’	=	K41	K42	K43	K44	K45	K46	K47	K48	V2’
P3’		K51	K52	K53	K54	K55	K56	K57	K58	U3’
Q3’		K61	K62	K63	K64	K65	K66	K67	K68	V3’
P4’		K71	K72	K73	K74	K75	K76	K77	K78	U4’
Q4’		K81	K82	K83	K84	K85	K86	K87	K88	V4’

Location matrix

0
0
1
2
3
4
0
0

②Total stiffness matrix



		U2	V2	U4	V4	
P2		K33	K34	K35	K36	U2
Q2	=	K43	K44	K45	K46	V2
P4		K53	K54	K55	K56	U4
Q4		K63	K64	K65	K66	V4

Freedom vector

$\{F\} =$

1_X	0
1_Y	0
2_X	1
2_Y	2
3_X	0
3_Y	0
4_X	3
4_Y	4

Node number

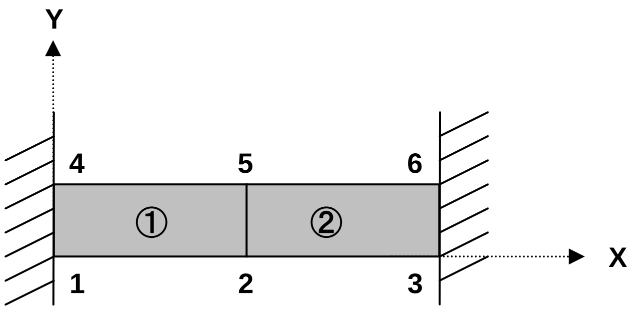
1	>	1'
2	>	2'
3	>	4'
4	>	3'

Location matrix

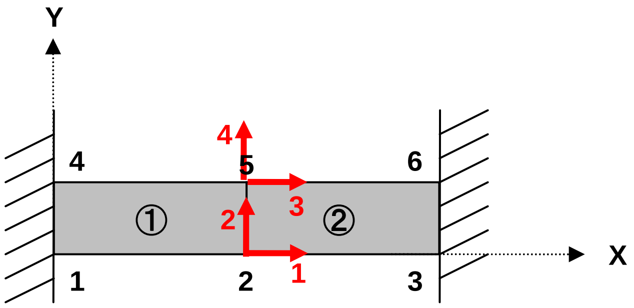
$1'_X$	0
$1'_Y$	0
$2'_X$	1
$2'_Y$	2
$3'_X$	3
$3'_Y$	4
$4'_X$	0
$4'_Y$	0

Exercise 2)

Please obtain the location matrix of the element ② of the following structure.



Answers 1) and 2)

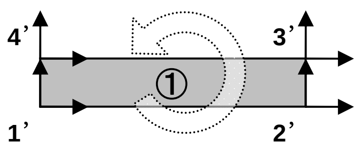


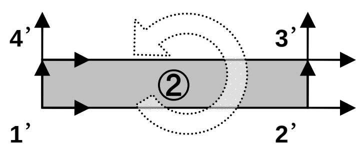
1) Freedom vector

{ F } =

	Initial		Restrained freedom = 1		Numbering
1_X	0		1		0
1_Y	0		1		0
2_X	0		0		1
2_Y	0	→	0	→	2
3_X	0		1		0
3_Y	0		1		0
4_X	0		1		0
4_Y	0		1		0
5_X	0		0		3
5_Y	0		0		4
6_X	0		1		0
6_Y	0		1		0

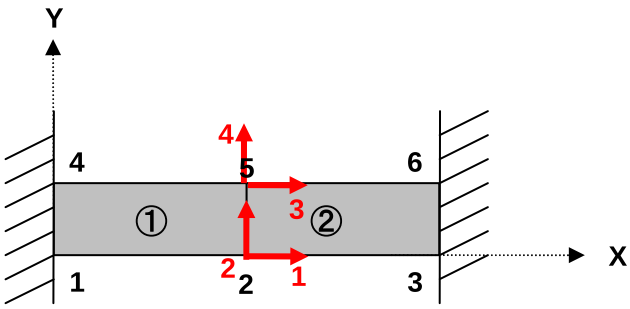
2) Location matrix





$1'_X$	0
$1'_Y$	0
$2'_X$	1
$2'_Y$	2
$3'_X$	3
$3'_Y$	4
$4'_X$	0
$4'_Y$	0

$1'_X$	1
$1'_Y$	2
$2'_X$	0
$2'_Y$	0
$3'_X$	0
$3'_Y$	0
$4'_X$	3
$4'_Y$	4

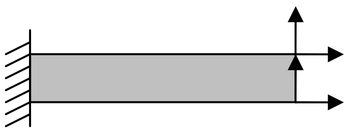


Element①

		1'x	1'y	2'x	2'y	3'x	3'y	4'x	4'y	
P1'		K11	K12	K13	K14	K15	K16	K17	K18	U1'
Q1'		K21	K22	K23	K24	K25	K26	K27	K28	V1'
P2'		K31	K32	K33	K34	K35	K36	K37	K38	U2'
Q2'	=	K41	K42	K43	K44	K45	K46	K47	K48	V2'
P3'		K51	K52	K53	K54	K55	K56	K57	K58	U3'
Q3'		K61	K62	K63	K64	K65	K66	K67	K68	V3'
P4'		K71	K72	K73	K74	K75	K76	K77	K78	U4'
Q4'		K81	K82	K83	K84	K85	K86	K87	K88	V4'

Location matrix

0
0
1
2
3
4
0
0



		U2	V2	U5	V5	
P2		K33	K34	K35	K36	U2
Q2	=	K43	K44	K45	K46	V2
P5		K53	K54	K55	K56	U5
Q5		K63	K64	K65	K66	V5

Element②

		$1'_X$	$1'_Y$	$2'_X$	$2'_Y$	$3'_X$	$3'_Y$	$4'_X$	$4'_Y$	
P1'		K11	K12	K13	K14	K15	K16	K17	K18	U1'
Q1'		K21	K22	K23	K24	K25	K26	K27	K28	V1'
P2'		K31	K32	K33	K34	K35	K36	K37	K38	U2'
Q2'	=	K41	K42	K43	K44	K45	K46	K47	K48	V2'
P3'		K51	K52	K53	K54	K55	K56	K57	K58	U3'
Q3'		K61	K62	K63	K64	K65	K66	K67	K68	V3'
P4'		K71	K72	K73	K74	K75	K76	K77	K78	U4'
Q4'		K81	K82	K83	K84	K85	K86	K87	K88	V4'

Location matrix

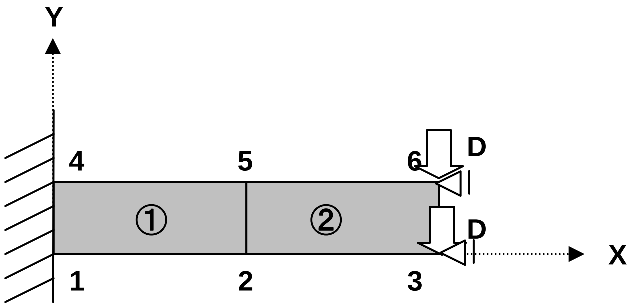
1
2
0
0
0
0
3
4

		U2	V2	U5	V5	
P2		K11	K12	K17	K18	U2
Q2	=	K21	K22	K27	K28	V2
P5		K71	K72	K77	K78	U5
Q5		K81	K82	K87	K88	V5

Total stiffness matrix

		U2	V2	U5	V5	
P2		$K33^{1}+K11^{2}$	$K34^{1}+K12^{2}$	$K35^{1}+K17^{2}$	$K36^{1}+K18^{2}$	U2
Q2	=	$K43^{1}+K21^{2}$	$K44^{1}+K22^{2}$	$K45^{1}+K27^{2}$	$K46^{1}+K28^{2}$	V2
P5		$K53^{1}+K71^{2}$	$K54^{1}+K72^{1}$	$K55^{1}+K77^{1}$	$K56^{1}+K78^{1}$	U5
Q5		$K63^{1}+K81^{2}$	$K64^{1}+K82^{2}$	$K65^{1}+K87^{1}$	$K66^{1}+K88^{2}$	V5

In case of displacement control

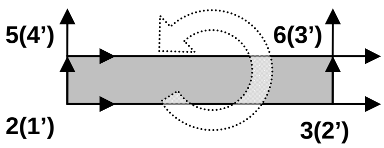


Treating the control direction 3_Y and 6_Y as restrained freedom, the freedom vector will be the same as the previous example.

$$\{F\}=\begin{array}{c|c|}1_X&0\\1_Y&0\\2_X&0\\2_Y&0\\3_X&0\\3_Y&0\\4_X&0\\4_Y&0\\5_X&0\\5_Y&0\\6_X&0\\6_Y&0\end{array}\rightarrow\begin{array}{|c|}1\\1\\0\\0\\1\\1\\1\\1\\0\\0\\1\\1\end{array}\rightarrow\begin{array}{|c|}0\\0\\1\\2\\0\\0\\0\\0\\3\\4\\0\\0\end{array},\quad\{F_P\}=\begin{array}{c|c|}1_X&0\\1_Y&0\\2_X&0\\2_Y&0\\3_X&0\\3_Y&1\\4_X&0\\4_Y&0\\5_X&0\\5_Y&0\\6_X&0\\6_Y&2\end{array}$$

Freedom vector Control vector

Element②



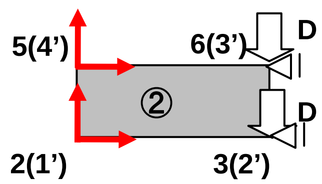
$$\{F\}=\begin{array}{c|c|}1'_X&1\\1'_Y&2\\2'_X&0\\2'_Y&0\\3'_X&0\\3'_Y&0\\4'_X&3\\4'_Y&4\end{array},\quad\{F_P\}=\begin{array}{c|c|}1'_X&0\\1'_Y&0\\2'_X&0\\2'_Y&1\\3'_X&0\\3'_Y&2\\4'_X&0\\4'_Y&0\end{array}$$

Location matrix Location of control

Element②

		1'X	1'Y	2'X	2'Y	3'X	3'Y	4'X	4'Y	
P1'		K11	K12	K13	K14	K15	K16	K17	K18	U1'
Q1'		K21	K22	K23	K24	K25	K26	K27	K28	V1'
P2'		K31	K32	K33	K34	K35	K36	K37	K38	U2'
Q2'	=	K41	K42	K43	K44	K45	K46	K47	K48	V2'
P3'		K51	K52	K53	K54	K55	K56	K57	K58	U3'
Q3'		K61	K62	K63	K64	K65	K66	K67	K68	V3'
P4'		K71	K72	K73	K74	K75	K76	K77	K78	U4'
Q4'		K81	K82	K83	K84	K85	K86	K87	K88	V4'

Location matrix	Location of control
1	0
2	0
0	0
0	1
0	0
0	2
3	0
4	0



		U2	V2	U5	V5	
P2		K11	K12	K17	K18	U2
Q2	=	K21	K22	K27	K28	V2
P5		K71	K72	K77	K78	U5
Q5		K81	K82	K87	K88	V5

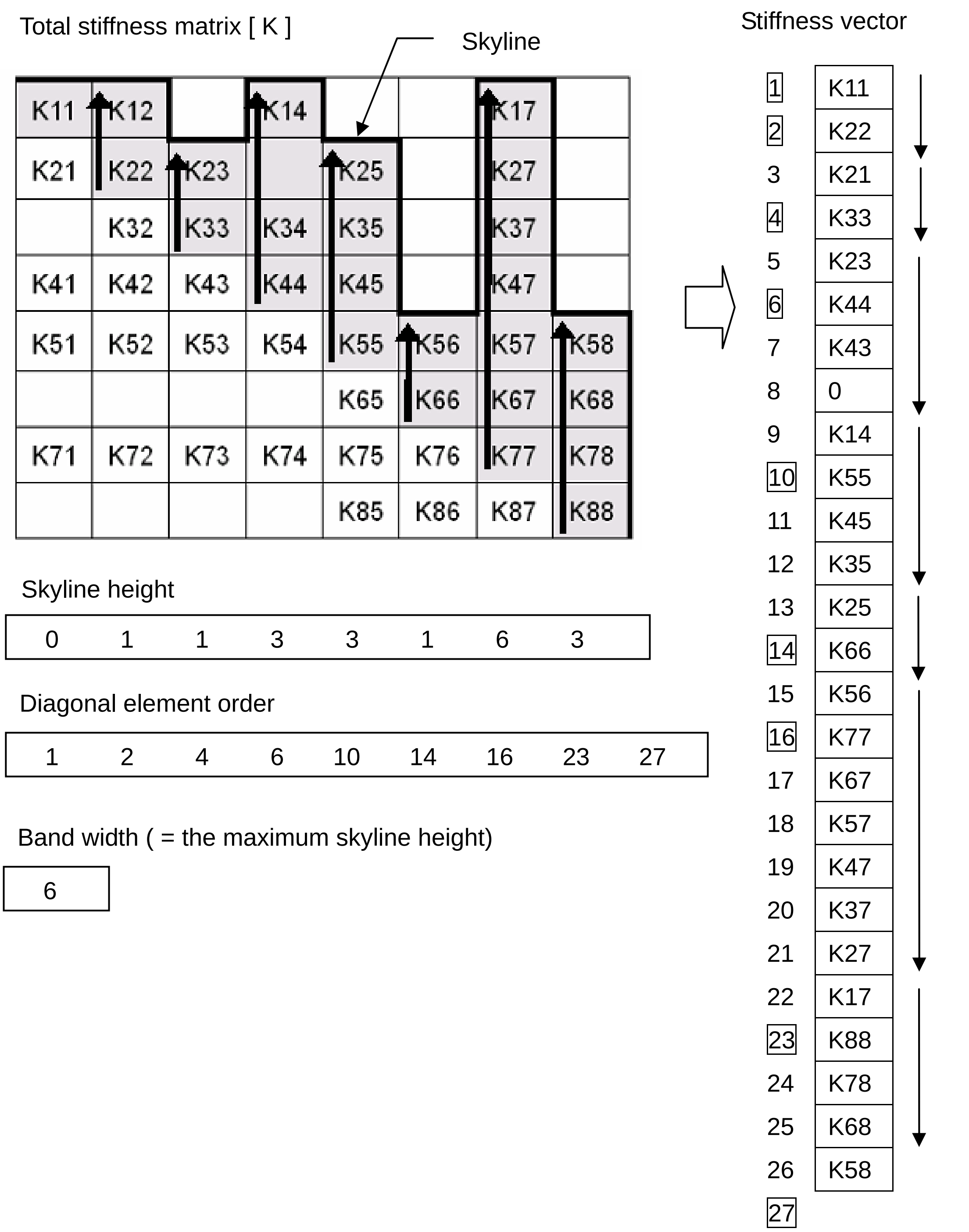
		V3	V6		
P2		K14	K16		
Q2	=	K24	K26	-1	D
P5		K74	K76	-1	
Q5		K84	K86		

Total equilibrium equation is

			U2	V2	U5	V5	
-K14-K16			$K33^1+K11^2$	$K34^1+K12^2$	$K35^1+K17^2$	$K36^1+K18^2$	U2
-K24-K26	D	=	$K43^1+K21^2$	$K44^1+K22^2$	$K45^1+K27^2$	$K46^1+K28^2$	V2
-K74-K76			$K53^1+K71^2$	$K54^1+K72^1$	$K55^1+K77^1$	$K56^1+K78^1$	U5
-K84-K86			$K63^1+K81^2$	$K64^1+K82^2$	$K65^1+K87^1$	$K66^1+K88^2$	V5

14. SKYLINE METHOD

Usually, the total stiffness matrix [K] is symmetric and sparse as shown below. Therefore, to save memory size and to reduce calculation time for linear equation solver, the elements in the upper triangular part of the matrix under the **Skyline** (thick line) are stored in an one-dimension vector.



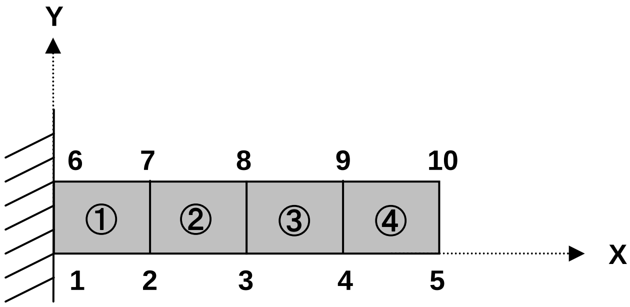
Exercise 3)

Using the same structure in Exercise 1), please compare

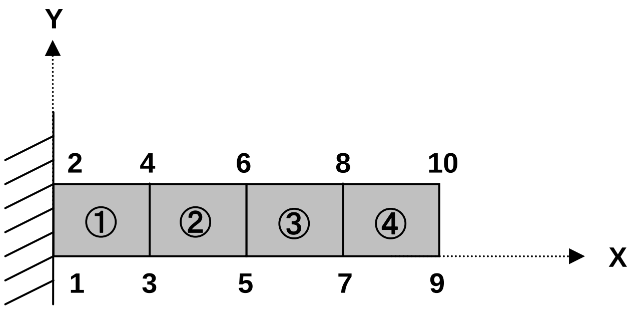
1) skyline height,
2) diagonal element order and
3) band width

between the following two cases with different node order.

Case 1

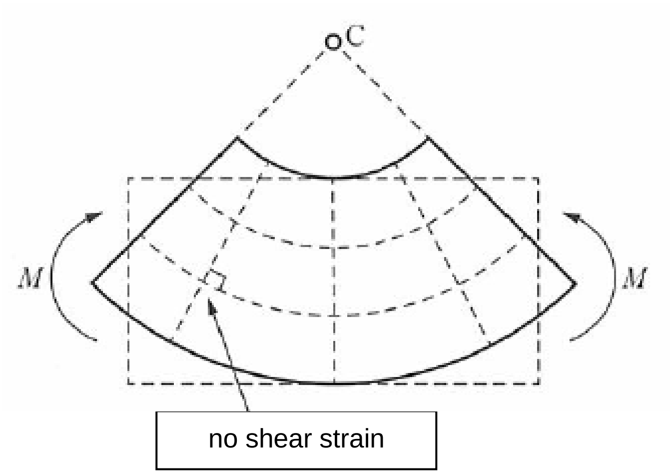


Case 2

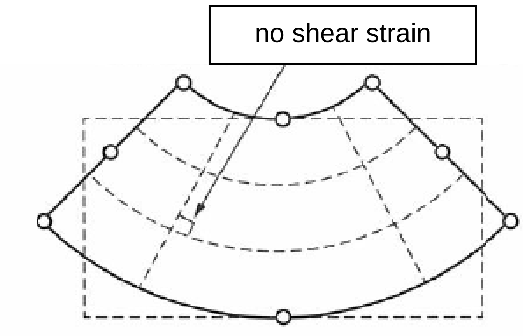


15. INCOMPATIBLE ELEMENT

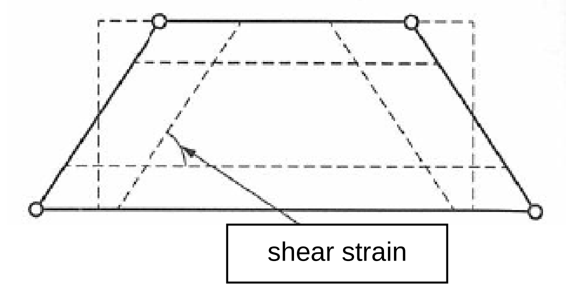
In an ideal situation, a beam under a pure bending moment experiences a curved shape change. The angle between the curved horizontal dotted line and the straight vertical line remains at 90 degree after bending. Therefore, no shear strain occurs inside a material.



To model the ideal shape change, an element should have the ability to assume the curved shape. The second order element with eight nodes enables to represent the curved shape. On the contrary, the first order element is not able to bend to curves and all dotted lines remain straight and the angle is no longer at 90 degree. To cause the angle to change under pure moment, an incorrect artificial shear strain and stress have been introduced. Therefore, the strain energy of the element is larger than ideal situation. As we demonstrated in the principle of virtual work method, overestimate of strain energy causes overestimate of stiffness matrix. This is the reason that the first order element with four nodes becomes overly stiff under the bending moment. This problem is called **shear locking**.



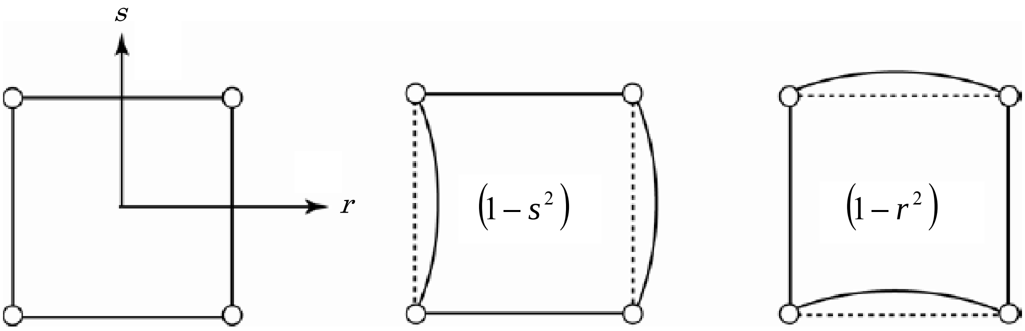
(a) Second order element



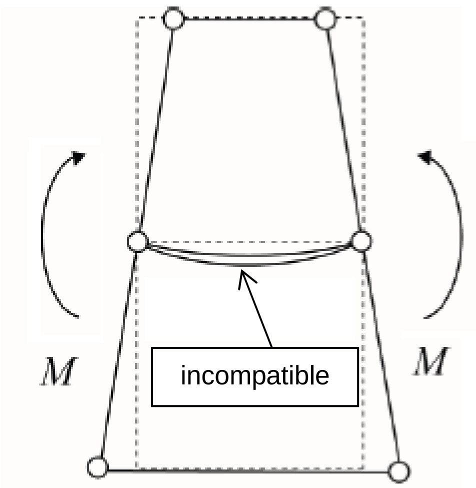
(b) First order element

To solve the problem in the first order element, we introduce the new displacement shape functions to add curved displacement modes.

$$u(r,s) = \sum_{i=1}^{4} h_i(r,s)u_i + (1-r^2)\alpha_1 + (1-s^2)\alpha_2$$
$$v(r,s) = \sum_{i=1}^{4} h_i(r,s)v_i + (1-r^2)\alpha_3 + (1-s^2)\alpha_4 \quad (15\text{-}1)$$



It can avoid over stiff in bending; however there might be incompatibility of deformation at the boundary. Therefore, this element is called **incompatible element**.



Equation (15-1) can be written in a matrix form as,

$$\mathbf{u}(r,s) = \mathbf{H}(r,s)\,\mathbf{U} + \mathbf{G}(r,s)\,\mathbf{A} \quad (15\text{-}2)$$

where,

$$H = \begin{bmatrix} h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 & 0 \\ 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 \end{bmatrix}, \quad G = \begin{bmatrix} g_1 & g_2 & 0 & 0 \\ 0 & 0 & g_1 & g_2 \end{bmatrix},$$

$$g_1 = (1-r^2), \quad g_2 = (1-s^2)$$

$$U^T = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4], \quad A^T = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4]$$

Stiffness matrix can be obtained from the "*Principle of Virtual Work Method*," which is expressed in the following form:

$$\int_V \bar{\varepsilon}^T \sigma \, dv = \bar{U}^T F \tag{15-3}$$

where, $\bar{\varepsilon}$ is a virtual strain vector, σ is a stress vector, $\bar{U}$ is a virtual displacement vector and F is a load vector, respectively.

The strain vector is calculated from the nodal displacement vector as,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial x} u_i + \dfrac{\partial g_1}{\partial x}\alpha_1 + \dfrac{\partial g_2}{\partial x}\alpha_2 \\ \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial y} v_i + \dfrac{\partial g_1}{\partial y}\alpha_3 + \dfrac{\partial g_2}{\partial y}\alpha_4 \\ \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial y} u_i + \dfrac{\partial g_1}{\partial y}\alpha_1 + \dfrac{\partial g_2}{\partial y}\alpha_2 + \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial x} v_i + \dfrac{\partial g_1}{\partial x}\alpha_3 + \dfrac{\partial g_2}{\partial x}\alpha_4 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\partial h_1}{\partial x} & 0 & \dfrac{\partial h_2}{\partial x} & 0 & \dfrac{\partial h_3}{\partial x} & 0 & \dfrac{\partial h_4}{\partial x} & 0 \\ 0 & \dfrac{\partial h_1}{\partial y} & 0 & \dfrac{\partial h_2}{\partial y} & 0 & \dfrac{\partial h_3}{\partial y} & 0 & \dfrac{\partial h_4}{\partial y} \\ \dfrac{\partial h_1}{\partial y} & \dfrac{\partial h_1}{\partial x} & \dfrac{\partial h_2}{\partial y} & \dfrac{\partial h_2}{\partial x} & \dfrac{\partial h_3}{\partial y} & \dfrac{\partial h_3}{\partial x} & \dfrac{\partial h_4}{\partial y} & \dfrac{\partial h_4}{\partial x} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} + \begin{pmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_2}{\partial x} & 0 & 0 \\ 0 & 0 & \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_2}{\partial y} \\ \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_2}{\partial y} & \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_2}{\partial x} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

$$\varepsilon = \boldsymbol{B}\boldsymbol{U} + \boldsymbol{G}\boldsymbol{A} \tag{15-4}$$

In the plane stress problem, the stress-strain relationship is expressed as,

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{15-5}$$

$$\sigma = \boldsymbol{D} \quad \varepsilon$$

Substituting Equation (15-4) into Equation (15-5),

$$\sigma = \boldsymbol{D}(\boldsymbol{B}\boldsymbol{U} + \boldsymbol{G}\boldsymbol{A}) \tag{15-6}$$

From the Principle of Virtual Work Method,

$$
\begin{aligned}
&\int_V \left(B\overline{U} + G\overline{A}\right)^T D\left(BU + GA\right)dv \\
&= \overline{U}^T\left(\int_V B^T DBdv\right)U + \overline{U}^T\left(\int_V B^T DGdv\right)A + \overline{A}^T\left(\int_V G^T DBdv\right)U + \overline{A}^T\left(\int_V G^T DGdv\right)A \\
&= \overline{U}^T F
\end{aligned}
\tag{15-7}
$$

It can be written in a Matrix form as;

$$
\begin{bmatrix} \overline{U}^T & \overline{A}^T \end{bmatrix}
\begin{bmatrix} \int_V B^T DBdv & \int_V B^T DGdv \\ \int_V G^T DBdv & \int_V G^T DGdv \end{bmatrix}
\begin{bmatrix} U \\ A \end{bmatrix}
= \begin{bmatrix} \overline{U}^T & \overline{A}^T \end{bmatrix}
\begin{bmatrix} F \\ 0 \end{bmatrix}
$$

Therefore, the equilibrium equation is obtained as,

$$
\begin{bmatrix} K_{UU} & K_{UA} \\ K_{AU} & K_{AA} \end{bmatrix}
\begin{bmatrix} U \\ A \end{bmatrix}
= \begin{bmatrix} F \\ 0 \end{bmatrix}
\tag{15-8}
$$

where

$$
\begin{aligned}
K_{UU} &= \int_V B^T DBdv, \quad K_{UA} = \int_V B^T DGdv \\
K_{AU} &= \int_V G^T DBdv, \quad K_{AA} = \int_V G^T DGdv
\end{aligned}
$$

From the second equation in (15-8), we can eliminate the incompatible displacement modes as;

$$
\begin{aligned}
K_{AU}U + K_{AA}A &= 0 \\
A &= -K_{AA}^{-1}K_{AU}U
\end{aligned}
\tag{15-9}
$$

Then, the element stiffness matrix is given by:

$$
\begin{aligned}
F &= KU \\
K &= K_{UU} - K_{UA}K_{AA}^{-1}K_{AU}
\end{aligned}
\tag{15-10}
$$

1) Evaluation of Jacobian Matrix

$$J = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial r} x_i & \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial r} y_i \\ \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial s} x_i & \sum_{i=1}^{4} \dfrac{\partial h_i}{\partial s} y_i \end{pmatrix} \tag{15-11}$$

2) Evaluation of the matrix G

From Equation (15-4),

$$G = \begin{pmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_2}{\partial x} & 0 & 0 \\ 0 & 0 & \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_2}{\partial y} \\ \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_2}{\partial y} & \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_2}{\partial x} \end{pmatrix} \tag{15-12}$$

The derivatives $\dfrac{\partial g_1}{\partial x}, \dfrac{\partial g_2}{\partial x}, \dfrac{\partial g_1}{\partial y}, \dfrac{\partial g_2}{\partial y}$ are calculated as,

$$\frac{\partial g_1}{\partial x} = \frac{\partial g_1}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g_1}{\partial s}\frac{\partial s}{\partial x}, \quad \frac{\partial g_2}{\partial x} = \frac{\partial g_2}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g_2}{\partial s}\frac{\partial s}{\partial x},$$
$$\frac{\partial g_1}{\partial y} = \frac{\partial g_1}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g_1}{\partial s}\frac{\partial s}{\partial y}, \quad \frac{\partial g_2}{\partial y} = \frac{\partial g_2}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g_2}{\partial s}\frac{\partial s}{\partial y}$$

In a matrix form,

$$\begin{pmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_2}{\partial x} \\ \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial s}{\partial x} \\ \dfrac{\partial r}{\partial y} & \dfrac{\partial s}{\partial y} \end{pmatrix} \begin{pmatrix} \dfrac{\partial g_1}{\partial r} & \dfrac{\partial g_2}{\partial r} \\ \dfrac{\partial g_1}{\partial s} & \dfrac{\partial g_2}{\partial s} \end{pmatrix} = J^{-1} \begin{pmatrix} \dfrac{\partial g_1}{\partial r} & \dfrac{\partial g_2}{\partial r} \\ \dfrac{\partial g_1}{\partial s} & \dfrac{\partial g_2}{\partial s} \end{pmatrix} \tag{15-13}$$

3) Evaluation of partial derivatives of the interpolation functions

$$\begin{aligned} \frac{\partial g_1}{\partial r} &= -2r, & \frac{\partial g_1}{\partial s} &= 0 \\ \frac{\partial g_2}{\partial r} &= 0 & \frac{\partial g_2}{\partial s} &= -2s \end{aligned} \tag{15-14}$$

4) Numerical integration

Using the 3 points Gaussian quadrature rule, the stiffness matrices are calculated numerically as follows:

$$K_{UU} = t\sum_{i=1}^{3}\sum_{j=1}^{3} w_i w_j F_{UU}(r_i, s_j)$$

$$K_{UA} = t\sum_{i=1}^{3}\sum_{j=1}^{3} w_i w_j F_{UA}(r_i, s_j)$$

$$K_{AU} = t\sum_{i=1}^{3}\sum_{j=1}^{3} w_i w_j F_{AU}(r_i, s_j)$$

$$K_{AA} = t\sum_{i=1}^{3}\sum_{j=1}^{3} w_i w_j F_{UA}(r_i, s_j) \qquad (15\text{-}15)$$

where

$$F_{UU}(r,s) = B(x(r,s), y(r,s))^T DB(x(r,s), y(r,s))\frac{\partial(x,y)}{\partial(r,s)}$$

$$F_{UA}(r,s) = B(x(r,s), y(r,s))^T DG(x(r,s), y(r,s))\frac{\partial(x,y)}{\partial(r,s)}$$

$$F_{AU}(r,s) = G(x(r,s), y(r,s))^T DB(x(r,s), y(r,s))\frac{\partial(x,y)}{\partial(r,s)}$$

$$F_{AA}(r,s) = G(x(r,s), y(r,s))^T DG(x(r,s), y(r,s))\frac{\partial(x,y)}{\partial(r,s)}$$

$$w_1 = 5/9 = 0.5556, \quad w_2 = 8/9 = 0.8889, \quad w_3 = 5/9 = 0.5556$$

$$r_1 = s_1 = -\sqrt{3/5} = -0.7746, \quad r_2 = s_2 = 0, \quad r_3 = s_3 = \sqrt{3/5} = 0.7746$$

Then, the element stiffness matrix is calculated as,

$$F = KU$$
$$K = K_{UU} - K_{UA}K_{AA}^{-1}K_{AU} \qquad (15\text{-}16)$$

5) Assemble of finite element

To total stiffness matrix can be obtained to assemble the element stiffness matrix over the areas of all finite elements.

$$K = \sum_{m} K_m \qquad (15\text{-}17)$$

where m denotes the m-th element.

6) Strain and Stress at Gaussian point

Strain at Gaussian point is obtained from Equations (15-4) and (15-9) as:

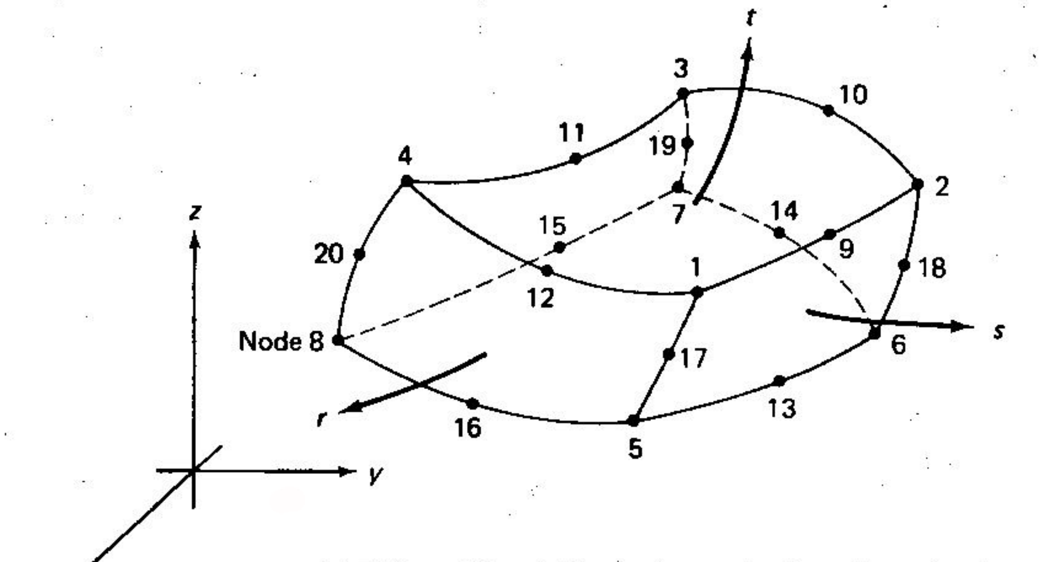
$$\varepsilon = BU + GA = (B - GK_{AA}^{-1}K_{AU})U \tag{15-8}$$

Stress at Gaussian point is obtained as:

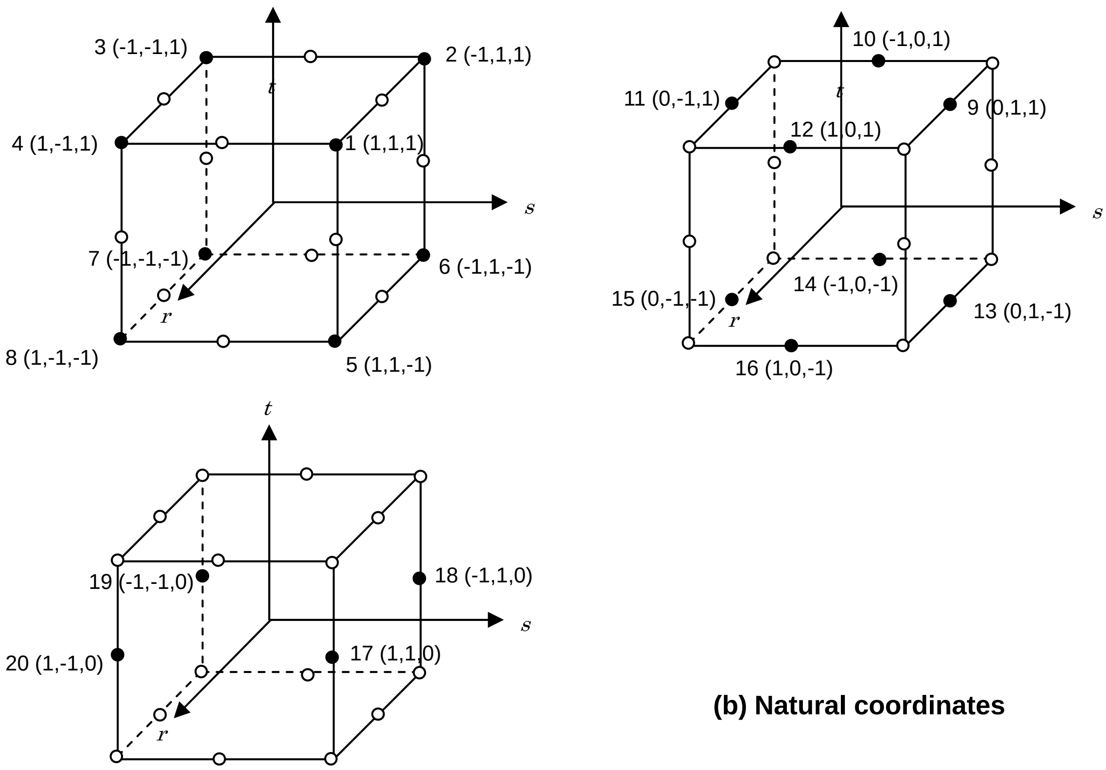
$$\sigma = D\varepsilon = (DB - DGK_{AA}^{-1}K_{AU})U \tag{15-9}$$

16. HEXAHEDRON ELEMENT

A hexahedron element is extensively used in modeling three-dimensional solids. It has eight corners, twelve edges, and six faces. By introducing natural coordinates (r, s, t), which vary from -1 to +1, it is possible to use the Gauss integration formulas.



(a) Eight to 20 variable-number-nodes three-dimensional element



(b) Natural coordinates

The corner numbering rule is described as follows to guarantee a positive Jacobian determinant.

1) Chose starting corner (number 1) and select one face pertaining to that corner.
2) Number the other corners on the face in counterclockwise order (number 2, 3, 4).
3) Number the corners of the opposite face to be opposite 1,2,3,4 as 5,6,7,8.
4) Number the middle point on the edge following the number of corner.

The interpolation function is given by

$$
\begin{aligned}
h_i(r,s,t) = & \frac{1}{8} r_i^2 s_i^2 t_i^2 (1 + r_i r)(1 + s_i s)(1 + t_i t)(r_i r + s_i s + t_i t - 2) \\
& + \frac{1}{4}\left(1 - r_i^2\right)(1 + s_i s)(1 + t_i t)\left(1 - r^2\right) \\
& + \frac{1}{4}\left(1 - s_i^2\right)(1 + t_i t)(1 + r_i r)\left(1 - s^2\right) \\
& + \frac{1}{4}\left(1 - t_i^2\right)(1 + r_i r)(1 + s_i s)\left(1 - t^2\right)
\end{aligned}
\tag{16-1}
$$

where

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
r_i	1	−1	−1	1	1	−1	−1	1	0	−1	0	1	0	−1	0	1	1	−1	−1	1
s_i	1	1	−1	−1	1	1	−1	−1	1	0	−1	0	1	0	−1	0	1	1	−1	−1
t_i	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	0	0	0	0

(16-2)

The partial derivative of the interpolation function is

$$
\begin{aligned}
\frac{\partial h_i(r,s,t)}{\partial r} = & \frac{1}{8} r_i^2 s_i^2 t_i^2 r_i (2 r_i r + s_i s + t_i t - 1)(1 + s_i s)(1 + t_i t) \\
& - \frac{1}{2}\left(1 - r_i^2\right)(1 + s_i s)(1 + t_i t) r + \frac{1}{4} r_i \left(1 - s_i^2\right)(1 + t_i t)\left(1 - s^2\right) + \frac{1}{4} r_i \left(1 - t_i^2\right)(1 + s_i s)\left(1 - t^2\right) \\
\frac{\partial h_i(r,s,t)}{\partial s} = & \frac{1}{8} r_i^2 s_i^2 t_i^2 s_i (r_i r + 2 s_i s + t_i t - 1)(1 + r_i r)(1 + t_i t) \\
& + \frac{1}{4} s_i \left(1 - r_i^2\right)(1 + t_i t)\left(1 - r^2\right) - \frac{1}{2}\left(1 - s_i^2\right)(1 + t_i t)(1 + r_i r) s + \frac{1}{4} s_i \left(1 - t_i^2\right)(1 + r_i r)\left(1 - t^2\right) \\
\frac{\partial h_i(r,s,t)}{\partial t} = & \frac{1}{8} r_i^2 s_i^2 t_i^2 r_i (r_i r + s_i s + 2 t_i t - 1)(1 + r_i r)(1 + s_i s) \\
& + \frac{1}{4} t_i \left(1 - r_i^2\right)(1 + s_i s)\left(1 - r^2\right) + \frac{1}{4} t_i \left(1 - s_i^2\right)(1 + r_i r)\left(1 - s^2\right) - \frac{1}{2} t_i \left(1 - t_i^2\right)(1 + r_i r)(1 + s_i s) t
\end{aligned}
\tag{16-3}
$$

The coordinate transfer function {x, y, z} is expressed using the interpolation functions

$$
\begin{aligned}
x(r,s,t) &= \sum_{i=1}^{20} h_i(r,s,t)x_i \\
y(r,s,t) &= \sum_{i=1}^{20} h_i(r,s,t)y_i \\
z(r,s,t) &= \sum_{i=1}^{20} h_i(r,s,t)z_i
\end{aligned}
\qquad (16\text{-}4)
$$

The deformation function {u, v, w} is expressed using the same interpolation functions.

$$
\begin{aligned}
u(r,s,t) &= \sum_{i=1}^{20} h_i(r,s,t)u_i \\
v(r,s,t) &= \sum_{i=1}^{20} h_i(r,s,t)v_i \\
w(r,s,t) &= \sum_{i=1}^{20} h_i(r,s,t)w_i
\end{aligned}
\qquad (16\text{-}5)
$$

In a matrix form,

$$
\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{bmatrix} h_1 & 0 & 0 & \cdots & h_{20} & 0 & 0 \\ 0 & h_1 & 0 & \cdots & 0 & h_{20} & 0 \\ 0 & 0 & h_1 & \cdots & 0 & 0 & h_{20} \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_{20} \\ v_{20} \\ w_{20} \end{pmatrix}
$$

$$
\boldsymbol{u(r,s,t) = H(r,s,t)\,U} \qquad (16\text{-}6)
$$

In case of the plane problem, the strain ε vector is defined as,

$$
\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{pmatrix}
= \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} \end{pmatrix}
= \begin{pmatrix} \sum_{i=1}^{20} \dfrac{\partial h_i}{\partial x} u_i \\ \sum_{i=1}^{20} \dfrac{\partial h_i}{\partial y} v_i \\ \sum_{i=1}^{20} \dfrac{\partial h_i}{\partial z} w_i \\ \sum_{i=1}^{20} \dfrac{\partial h_i}{\partial y} u_i + \sum_{i=1}^{20} \dfrac{\partial h_i}{\partial x} v_i \\ \sum_{i=1}^{20} \dfrac{\partial h_i}{\partial z} v_i + \sum_{i=1}^{20} \dfrac{\partial h_i}{\partial y} w_i \\ \sum_{i=1}^{20} \dfrac{\partial h_i}{\partial x} w_i + \sum_{i=1}^{20} \dfrac{\partial h_i}{\partial z} u_i \end{pmatrix}
= \begin{bmatrix}
\dfrac{\partial h_1}{\partial x} & 0 & 0 & \cdots & \dfrac{\partial h_{20}}{\partial x} & 0 & 0 \\
0 & \dfrac{\partial h_1}{\partial y} & 0 & \cdots & 0 & \dfrac{\partial h_{20}}{\partial y} & 0 \\
0 & 0 & \dfrac{\partial h_1}{\partial z} & \cdots & 0 & 0 & \dfrac{\partial h_{20}}{\partial z} \\
\dfrac{\partial h_1}{\partial y} & \dfrac{\partial h_1}{\partial x} & 0 & \cdots & \dfrac{\partial h_{20}}{\partial y} & \dfrac{\partial h_{20}}{\partial x} & 0 \\
0 & \dfrac{\partial h_1}{\partial z} & \dfrac{\partial h_1}{\partial y} & \cdots & 0 & \dfrac{\partial h_{20}}{\partial z} & \dfrac{\partial h_{20}}{\partial y} \\
\dfrac{\partial h_1}{\partial z} & 0 & \dfrac{\partial h_1}{\partial x} & \cdots & \dfrac{\partial h_{20}}{\partial z} & 0 & \dfrac{\partial h_{20}}{\partial x}
\end{bmatrix}
\begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_{20} \\ v_{20} \\ w_{20} \end{pmatrix}
$$

$$
\varepsilon = \boldsymbol{B\,U} \qquad (16\text{-}7)
$$

The stress-strain relationship is expressed as,

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}, \quad \varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_z}{E}, \quad \varepsilon_z = \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E},$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G}, \quad \gamma_{zx} = \frac{\tau_{zx}}{G}, \quad G = \frac{1}{2(1+\nu)}E \tag{16-8}$$

In a matrix form,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{pmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} \tag{16-9}$$

From the inverse matrix, the constitutive matrix is obtained as,

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} = \frac{2G}{1-2\nu} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{pmatrix}$$

$$\boldsymbol{\sigma} = \boldsymbol{D}\ \boldsymbol{\varepsilon} \tag{16-10}$$

Substituting Equation (16-7) into Equation (16-10),

$$\boldsymbol{\sigma} = \boldsymbol{D}\,\boldsymbol{B}\,\boldsymbol{U} \tag{16-11}$$

From the Principle of Virtual Work Method, the stiffness matrix is obtained as,

$$F = KU, \quad K = \int_V B^T DB dv \tag{16-12}$$

Since this integration is defined in x-y-z coordinate, we must transfer the coordinate into r-s-t coordinate to use the numerical integration method. Introducing the **Jacobian matrix**,

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{pmatrix} ; \textit{Jacobian Matrix} \tag{16-13}$$

The above integration is expressed in r-s coordinate as,

$$K = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} B(r,s,t)^T DB(r,s,t) \frac{\partial(x,y,z)}{\partial(r,s,t)} drdsdt \tag{16-14}$$

where

$$\frac{\partial(x,y,z)}{\partial(r,s,t)} = \det J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \tag{16-15}$$

1) Evaluation of Jacobian Matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{20} \frac{\partial h_i}{\partial r} x_i & \sum_{i=1}^{20} \frac{\partial h_i}{\partial r} y_i & \sum_{i=1}^{20} \frac{\partial h_i}{\partial r} z_i \\ \sum_{i=1}^{20} \frac{\partial h_i}{\partial s} x_i & \sum_{i=1}^{20} \frac{\partial h_i}{\partial s} y_i & \sum_{i=1}^{20} \frac{\partial h_i}{\partial s} z_i \\ \sum_{i=1}^{20} \frac{\partial h_i}{\partial t} x_i & \sum_{i=1}^{20} \frac{\partial h_i}{\partial t} y_i & \sum_{i=1}^{20} \frac{\partial h_i}{\partial t} z_i \end{pmatrix} \tag{16-16}$$

2) Evaluation of the matrix B

From Equation (16-7),

$$B = \begin{bmatrix} \frac{\partial h_1}{\partial x} & 0 & 0 & \cdots & \frac{\partial h_{20}}{\partial x} & 0 & 0 \\ 0 & \frac{\partial h_1}{\partial y} & 0 & \cdots & 0 & \frac{\partial h_{20}}{\partial y} & 0 \\ 0 & 0 & \frac{\partial h_1}{\partial z} & \cdots & 0 & 0 & \frac{\partial h_{20}}{\partial z} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial x} & 0 & \cdots & \frac{\partial h_{20}}{\partial y} & \frac{\partial h_{20}}{\partial x} & 0 \\ 0 & \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial y} & \cdots & 0 & \frac{\partial h_{20}}{\partial z} & \frac{\partial h_{201}}{\partial y} \\ \frac{\partial h_1}{\partial z} & 0 & \frac{\partial h_1}{\partial x} & \cdots & \frac{\partial h_{20}}{\partial z} & 0 & \frac{\partial h_{20}}{\partial x} \end{bmatrix} \tag{16-17}$$

The derivatives $\frac{\partial h_1}{\partial x},\cdots,\frac{\partial h_{20}}{\partial x},\frac{\partial h_1}{\partial y},\cdots,\frac{\partial h_{20}}{\partial y},\frac{\partial h_1}{\partial z},\cdots,\frac{\partial h_{20}}{\partial z}$ are calculated as,

$$\begin{aligned}
&\frac{\partial h_1}{\partial x}=\frac{\partial h_1}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial h_1}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial h_1}{\partial t}\frac{\partial t}{\partial x},\quad\cdots\quad,\frac{\partial h_{20}}{\partial x}=\frac{\partial h_{20}}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial h_{20}}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial h_{20}}{\partial t}\frac{\partial t}{\partial x},\\
&\frac{\partial h_1}{\partial y}=\frac{\partial h_1}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial h_1}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial h_1}{\partial t}\frac{\partial t}{\partial y},\quad\cdots\quad,\frac{\partial h_{20}}{\partial y}=\frac{\partial h_{20}}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial h_{20}}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial h_{20}}{\partial t}\frac{\partial t}{\partial y},\\
&\frac{\partial h_1}{\partial z}=\frac{\partial h_1}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial h_1}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial h_1}{\partial t}\frac{\partial t}{\partial z},\quad\cdots\quad,\frac{\partial h_{20}}{\partial z}=\frac{\partial h_{20}}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial h_{20}}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial h_{20}}{\partial t}\frac{\partial t}{\partial z}
\end{aligned}\qquad (16\text{-}18)$$

In a matrix form,

$$\begin{pmatrix}\frac{\partial h_1}{\partial x} & \cdots & \frac{\partial h_{20}}{\partial x}\\ \frac{\partial h_1}{\partial y} & \cdots & \frac{\partial h_{20}}{\partial y}\\ \frac{\partial h_1}{\partial z} & \cdots & \frac{\partial h_{20}}{\partial z}\end{pmatrix}=\begin{pmatrix}\frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} & \frac{\partial t}{\partial x}\\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} & \frac{\partial t}{\partial y}\\ \frac{\partial r}{\partial z} & \frac{\partial s}{\partial z} & \frac{\partial t}{\partial z}\end{pmatrix}\begin{pmatrix}\frac{\partial h_1}{\partial r} & \cdots & \frac{\partial h_{20}}{\partial r}\\ \frac{\partial h_1}{\partial s} & \cdots & \frac{\partial h_{20}}{\partial s}\\ \frac{\partial h_1}{\partial t} & \cdots & \frac{\partial h_{20}}{\partial t}\end{pmatrix}=J^{-1}\begin{pmatrix}\frac{\partial h_1}{\partial r} & \cdots & \frac{\partial h_{20}}{\partial r}\\ \frac{\partial h_1}{\partial s} & \cdots & \frac{\partial h_{20}}{\partial s}\\ \frac{\partial h_1}{\partial t} & \cdots & \frac{\partial h_{20}}{\partial t}\end{pmatrix}\qquad (16\text{-}19)$$

3) Stress and strain at Gaussian point

If you use the 3-points Gaussian Integration Method, there are 27 Gaussian points $(r_i, s_j, t_k)(i=1,2,3,\ j=1,2,3,\ k=1,2,3)$ in an element. The stress and strain at the Gaussian point (r_i, s_j, t_k) is obtained from

$$\sigma_{ij}=\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{pmatrix}_{ij}=DB_{ij}U\ ,\qquad \varepsilon_{ij}=\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{pmatrix}_{ij}=B_{ij}U\qquad (16\text{-}20)$$

4) Numerical integration

To reduce calculation time, the 2 points Gaussian quadrature rule is adopted to calculate the stiffness matrix as follows:

$$K = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} B(r,s,t)^T DB(r,s,t)\frac{\partial(x,y,z)}{\partial(r,s,t)}drdsdt$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} F(r,s,t)drdsdt \qquad (16\text{-}21)$$

$$= \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} w_i w_j w_k F(r_i, s_j, t_k)$$

where

$$F(r,s,t) = B(r,s,t)^T DB(r,s,t)\frac{\partial(x,y,z)}{\partial(r,s,t)}$$

$$w_1 = w_2 = 1.0$$

$$r_1 = s_1 = t_1 = -\sqrt{1/3} = -0.57735, \quad r_2 = s_2 = t_2 = \sqrt{1/3} = 0.57735$$

5) Assemble of finite element

To total stiffness matrix can be obtained to assemble the element stiffness matrix over the areas of all finite elements.

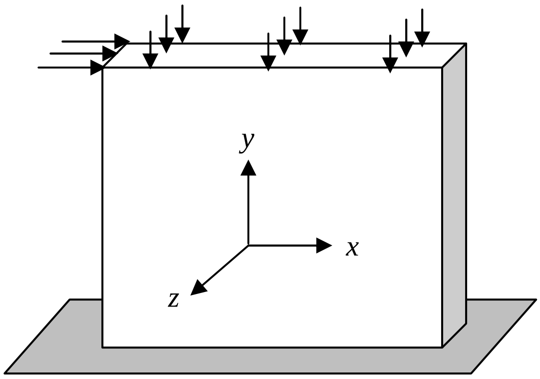
$$K = \sum_m K_m \qquad (16\text{-}22)$$

where m denotes the m-th element.

17. PLAIN STRESS AND PLAIN STRAIN

1) Plain stress assumption

If a thin plate in the x-y coordinate is loaded by forces at the boundary and distributed uniformly over the thickness, the stress components along z-axis, $\sigma_z, \tau_{xz}, \tau_{yz}$, are zero on both faces of the plate. It can be assumed that they are also zero within the plate.



Plain stress condition

Then, from Equation (16-9), the stress-strain relationship is expressed as,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{pmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z = 0 \\ \tau_{xy} \\ \tau_{yz} = 0 \\ \tau_{zx} = 0 \end{pmatrix} \tag{17-1}$$

Thus,

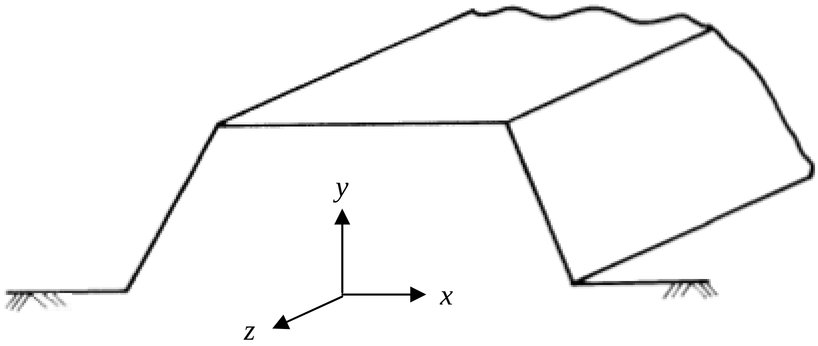
$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{bmatrix} \frac{1}{E} & -\nu\frac{1}{E} & 0 \\ -\nu\frac{1}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \tag{17-2}$$

or

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{17-3}$$

2) Plain strain assumption

When the dimension of the body in z-direction is very large such as a retaining wall with lateral pressure, it may be assumed that the displacement in the z-direction is prevented.



Plain strain condition

Since the longitudinal displacement is zero, from Equation (16-10),

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} = \frac{2G}{1-2\nu} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z = 0 \\ \gamma_{xy} \\ \gamma_{yz} = 0 \\ \gamma_{zx} = 0 \end{pmatrix} \quad (17\text{-}4)$$

Thus,

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{2G}{1-2\nu} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \quad (17\text{-}5)$$

or

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \frac{1}{2G} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \quad (17\text{-}6)$$

Note that, from Equation (17-4), the normal stress along z-axis is not zero.

$$\sigma_z \neq 0$$

which will be considered in case of nonlinear problem.

CHAPTER 2

DYNAMIC ANALYSIS

Index of Chapter 2

1. MASS MATRIX FOR ISOPARAMETRIC ELEMENT

1) Formulation

Under dynamic loading, the "*Principle of Virtual Work Method in dynamic problem = D'Alembert's principle*" is expressed in the following form:

$$\delta Q - \delta W = 0 \tag{1-1}$$

$$\delta Q = \int_V \bar{\varepsilon}^T \sigma \, dv \tag{1-2}$$

$$\delta W = \int_V \bar{u}^T \left(F - \rho \frac{\partial^2 u}{\partial t^2} - \mu \frac{\partial u}{\partial t} \right) dv \tag{1-3}$$

where F: body force, ρ: density, μ: damping coefficient

Substitute following relationships into above equations:

$$\boldsymbol{u(r,\ s) = H(r,\ s)\ U}$$
$$\boldsymbol{\varepsilon = B\ U}$$
$$\boldsymbol{\sigma = D\ B\ U}$$

$$\delta Q = \int_V \left(B\bar{U}\right)^T \left(DBU\right) dv = \bar{U}^T \left(\int_V B^T DB dv \right) U \tag{1-4}$$

$$\delta W = \int_V \bar{u}^T \left(F - \rho \frac{\partial^2 u}{\partial t^2} - \mu \frac{\partial u}{\partial t} \right) dv$$
$$= \bar{U}^T \int_V H^T F \, dv - \bar{U}^T \left(\int_V H^T \rho \, H dv \right) \ddot{U} - \bar{U}^T \left(\int_V H^T \mu \, H dv \right) \dot{U} \tag{1-5}$$

Therefore, from Equation (1-1),

$$\left(\int_V H^T \rho \, H dv \right) \ddot{U} + \left(\int_V H^T \mu \, H dv \right) \dot{U} + \left(\int_V B^T C B dv \right) U = \bar{U}^T \int_V H^T F \, dv \tag{1-5}$$

That is, the equilibrium equation is expressed as,

$$M\ddot{U} + C\dot{U} + KU = R$$

$$M = \left(\int_V H^T \rho \, H dv \right), \quad C = \left(\int_V H^T \mu \, H dv \right), \quad K = \left(\int_V B^T DB dv \right), \quad R = \int_V H^T F \, dv \tag{1-6}$$

2) Evaluation of the matrix H and H^TH

For a two dimensional isoparametric element,

$$H = \begin{bmatrix} h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 & 0 \\ 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 \end{bmatrix} \tag{1-7}$$

$$H^T H = \begin{bmatrix} h_1 & 0 \\ 0 & h_1 \\ h_2 & 0 \\ 0 & h_2 \\ h_3 & 0 \\ 0 & h_3 \\ h_4 & 0 \\ 0 & h_4 \end{bmatrix} \begin{bmatrix} h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 & 0 \\ 0 & h_1 & 0 & h_2 & 0 & h_3 & 0 & h_4 \end{bmatrix}$$

$$= \begin{bmatrix} h_1^2 & 0 & h_1h_2 & 0 & h_1h_3 & 0 & h_1h_4 & 0 \\ & h_1^2 & 0 & h_1h_2 & 0 & h_1h_3 & 0 & h_1h_4 \\ & & h_2^2 & 0 & h_2h_3 & 0 & h_2h_4 & 0 \\ & & & h_2^2 & 0 & h_2h_3 & 0 & h_2h_4 \\ & & & & h_3^2 & 0 & h_3h_4 & 0 \\ & & & & & h_3^2 & 0 & h_3h_4 \\ & sym. & & & & & h_4^2 & 0 \\ & & & & & & & h_4^2 \end{bmatrix} \tag{1-8}$$

For a three-dimensional hexahedron element,

$$H = \begin{bmatrix} h_1 & 0 & 0 & \cdots & h_{20} & 0 & 0 \\ 0 & h_1 & 0 & \cdots & 0 & h_{20} & 0 \\ 0 & 0 & h_1 & \cdots & 0 & 0 & h_{20} \end{bmatrix} \tag{1-9}$$

$$
H^T H = \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & h_1 \\ \vdots & \vdots & \vdots \\ h_{20} & 0 & 0 \\ 0 & h_{20} & 0 \\ 0 & 0 & h_{20} \end{bmatrix} \begin{bmatrix} h_1 & 0 & 0 & \cdots & h_{20} & 0 & 0 \\ 0 & h_1 & 0 & \cdots & 0 & h_{20} & 0 \\ 0 & 0 & h_1 & \cdots & 0 & 0 & h_{20} \end{bmatrix}
$$

$$
= \begin{bmatrix} h_1^2 & 0 & 0 & h_1 h_2 & 0 & 0 & \cdots & h_1 h_{20} & 0 & 0 \\ & h_1^2 & 0 & 0 & h_1 h_2 & 0 & \cdots & 0 & h_1 h_{20} & 0 \\ & & h_1^2 & 0 & 0 & h_1 h_2 & \cdots & 0 & 0 & h_1 h_{20} \\ & & & h_2^2 & 0 & 0 & \cdots & h_2 h_{20} & 0 & 0 \\ & & & & h_2^2 & 0 & \cdots & 0 & h_2 h_{20} & 0 \\ & & & & & h_2^2 & \cdots & 0 & 0 & h_2 h_{20} \\ & & & & & & \ddots & \vdots & \vdots & \vdots \\ & & sym. & & & & & h_{20}^2 & 0 & 0 \\ & & & & & & & & h_{20}^2 & 0 \\ & & & & & & & & & h_{20}^2 \end{bmatrix} \qquad (1\text{-}10)
$$

3) Numerical integration

The integration for mass matrix can be expressed in r-s coordinate as,

$$
\begin{aligned} M &= \int_V H^T \rho\, H dv \\ &= t \int_{V(x,y)} H^T \rho H dxdy \qquad (1\text{-}11) \\ &= t\rho \int_{-1}^{1}\int_{-1}^{1} H^T H \left(\det J\right) drds \end{aligned}
$$

The integration can be evaluated by the Gaussian Integration Formula as,

$$
\begin{aligned} M &= t\rho \sum_{i=1}^{3}\sum_{j=1}^{3} w_i w_j G\left(r_i, s_j\right) \\ G\left(r_i, s_j\right) &= H^T\left(r_i, s_j\right) H\left(r_i, s_j\right) \left(\det J\right)_{ij} \end{aligned} \qquad (1\text{-}12)
$$

4) Lumped mass model

The mass matrix obtained from the density of material is called the **consistent mass matrix** using the same interpolation functions for stiffness matrix, mass matrix and load vectors. Instead of performing the integrations, we may evaluate an approximate mass matrix by lumping equal parts of the total element mass to the nodal points which is called the **lumped mass matrix**. An important advantage of using a lumped mass matrix is that the matrix is diagonal and the numerical operations for the solution of the dynamic equations are reduced significantly.

Suppose that the consistent mass matrix is expressed as,

$$M_C = \begin{bmatrix} m_{C11} & m_{C12} & \cdots & m_{C1n} \\ m_{C21} & m_{C22} & \cdots & m_{C2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{Cn1} & m_{Cn2} & \cdots & m_{Cnn} \end{bmatrix} \tag{1-13}$$

The lumped matrix can be evaluated from the consistent mass matrix from the following formula:

$$M_L = \begin{bmatrix} m_{L11} & 0 & \cdots & 0 \\ 0 & m_{L22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_{Lnn} \end{bmatrix}, \quad m_{Lii} = \sum_{j=1}^{n} m_{Cij} \tag{1-14}$$

5) Gravity force

Gravity force is considered as a body force as

$$R = \int_V H^T F \, dv = t \int_{V(x,y)} H^T F dxdy = t \int_{-1}^{1} \int_{-1}^{1} H^T F \left(\det J\right) drds, \quad F = \begin{pmatrix} 0 \\ -\rho g \end{pmatrix} \tag{1-15}$$

where $g(=9.8m/s^2)$ is the gravity acceleration.

$$R = t \sum_{i=1}^{3} \sum_{j=1}^{3} w_i w_j I(r_i, s_j)$$
$$I(r_i, s_j) = H^T (r_i, s_j) \begin{pmatrix} 0 \\ -\rho g \end{pmatrix} (\det J)_{ij} \tag{1-16}$$

2. EIGEN VALUE PROBLEM

The free vibration equilibrium equation without damping is

$$M\ddot{U} + KU = 0 \tag{2-1}$$

where K is the stiffness matrix and M is the lumped mass matrix in the form,

$$M = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix} \tag{2-2}$$

The solution can be postulated to be in the form

$$U = \phi\, e^{i\omega t} \tag{2-3}$$

where ϕ is a vector of order n, ω is a frequency of vibration of the vector ϕ .

Then, the generalized eigenproblem is,

$$K\phi = \omega^2 M\phi \tag{2-4}$$

This eigenproblem yields the n eigensolutions $(\omega_1^2, \phi_1), (\omega_2^2, \phi_2), \cdots, (\omega_n^2, \phi_n)$ where the eigenvevtors are M-orthonormalized as,

$$\phi_i^T M\phi_j == \sum_{k=1}^{n} m_k \phi_{i,k}\phi_{j,k} = \begin{cases} = 1\,; & i = j \\ = 0\,; & i \neq j \end{cases} \tag{2-5}$$

$$0 \leq \omega_1^2 \leq \omega_2^2 \leq \cdots \leq \omega_n^2 \tag{2-6}$$

The vector ϕ_i is called the i-th mode shape vector, and ω_i is the corresponding frequency of vibration.

Defining a matrix Φ whose columns are the eigenvectors and a diagonal matrix Ω^2 which stores the eigenvalues on its diagonal as,

$$\Phi = [\phi_1 \quad \phi_2 \quad \cdots \quad \phi_n], \qquad \Omega^2 = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{bmatrix} \tag{2-7}$$

We introduce the following transformation on the displacement vector of the equilibrium Equation (1-6),

$$U(t) = \Phi X(t) \tag{2-8}$$

Then,

$$M\Phi\ddot{X} + C\Phi\dot{X} + K\Phi X = R \tag{2-9}$$

Multiplying Φ^T,

$$\Phi^T M\Phi\ddot{X} + \Phi^T C\Phi\dot{X} + \Phi^T K\Phi X = \Phi^T R \tag{2-10}$$

Using $\Phi^T M\Phi = I, \quad \Phi^T K\Phi = \Omega^2$,

$$\ddot{X} + \Phi^T C\Phi\dot{X} + \Omega^2 X = \Phi^T R \tag{2-11}$$

A damping matrix that is diagonalized by Φ is called a classical damping matrix.

$$\Phi^T C\Phi = \overline{C} = \begin{bmatrix} 2h_1\omega_1 & & & \\ & 2h_2\omega_2 & & \\ & & \ddots & \\ & & & 2h_n\omega_n \end{bmatrix} \tag{2-12}$$

where h_i is the modal damping ratio of the i-th mode.

Then, Equation (2-11) reduce to n- equations of the form

$$\ddot{x}_i(t) + 2h_i\omega_i\dot{x}(t) + \omega_i^{\,2}x(t) = r_i(t) \tag{2-13}$$

where $r_i(t) = \phi_i^{\,T} R(t)$

The initial conditions on X(t) are obtained from Equation (2-8) as,

$$X_{t=0} = \Phi^T M U_{t=0}, \quad \dot{X}_{t=0} = \Phi^T M \dot{U}_{t=0} \tag{2-14}$$

3. CLASSICAL DAMPING

Three procedures for constructing a classical damping matrix are described as follow:

1) Proportional damping

Consider first mass-proportional damping and stiffness-proportional damping,

$$C = a_0 M \quad \text{and} \quad C = a_1 K \tag{3-1}$$

where the constants a_0, a_1 have units of sec^{-1} and sec, respectively.

For a system with mass-proportional damping, the generalized damping for the i-th mode is,

$$c_i = a_0 m_i, \qquad c_i / m_i = 2h_i\omega_i \tag{3-2}$$

Therefore,

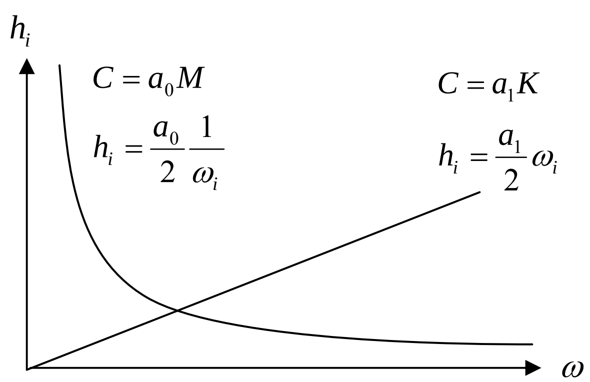
$$a_0 = 2h_i\omega_i, \qquad h_i = \frac{a_0}{2}\frac{1}{\omega_i} \tag{3-3}$$

Similarly, for a system with stiffness-proportional damping, the generalized damping for the i-th mode is,

$$c_i = a_1\omega_i^2 m_i, \qquad c_i / m_i = 2h_i\omega_i \tag{3-4}$$

Therefore,

$$a_1 = \frac{2h_i}{\omega_i}, \qquad h_i = \frac{a_1}{2}\omega_i \tag{3-5}$$



2) Rayleigh damping

A Rayleigh damping matrix is proportional to the mass and stiffness matrices as,

$$C = a_0 M + a_1 K \tag{3-6}$$

The modal damping ratio for the i-th mode of such a system is,

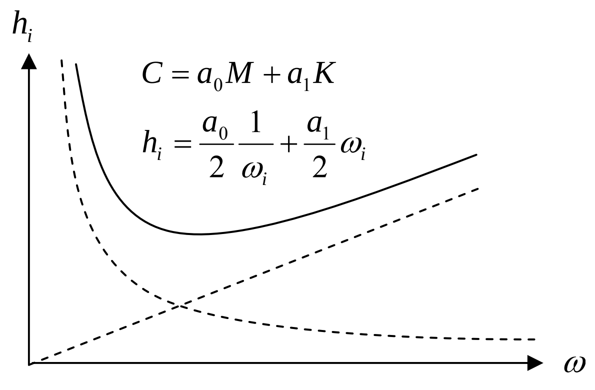
$$h_i = \frac{a_0}{2}\frac{1}{\omega_i} + \frac{a_1}{2}\omega_i \tag{3-7}$$

The coefficients a_0, a_1 can be determined from specified damping ratios h_1, h_2 modes, respectively. Expressing Equation (3-3) for these two modes in matrix form leads to:

$$\frac{1}{2}\begin{bmatrix} 1/\omega_1 & \omega_1 \\ 1/\omega_2 & \omega_2 \end{bmatrix}\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} h_1 \\ h_2 \end{Bmatrix} \tag{3-8}$$

Solving the above system, we obtain the coefficients a_0, a_1:

$$\begin{aligned} a_0 &= \frac{2\omega_1\omega_2(\omega_1 h_2 - \omega_2 h_1)}{(\omega_1{}^2 - \omega_2{}^2)} \\ a_1 &= \frac{2(\omega_1 h_2 - \omega_2 h_1)}{(\omega_1{}^2 - \omega_2{}^2)} \end{aligned} \tag{3-9}$$



3) Modal damping

It is an alternative procedure to determinate a classical damping matrix from modal damping ratios. From the definition of a classical damping matrix,

$$\Phi^T C \Phi = \overline{C} = \begin{bmatrix} 2h_1\omega_1 & & & \\ & 2h_2\omega_2 & & \\ & & \ddots & \\ & & & 2h_n\omega_n \end{bmatrix}$$

$$C = \left(\Phi^T\right)^{-1} \overline{C} \Phi^{-1} \tag{3-10}$$

Since $\Phi^T M \Phi = I$,

$$\left(\Phi^T\right)^{-1} = M\Phi, \quad \Phi^{-1} = \Phi^T M \tag{3-11}$$

Therefore,

$$C = (M\Phi)\overline{C}\left(\Phi^T M\right) \tag{3-12}$$

4. EQUATION OF MOTION UNDER EARTHQUAKE GROUND MOTION

1) Equation of motion under earthquake ground motion

Earthquake ground motions are defined as two components acceleration; $\ddot{X}_0$ and $\ddot{Y}_0$, in X and Y directions. The inertia forces at node i are defined as,

$$\begin{Bmatrix} \vdots \\ -M_i\left(\ddot{u}_i + \ddot{X}_0\right) \\ -M_i\left(\ddot{v}_i + \ddot{Y}_0\right) \\ \vdots \end{Bmatrix} = -M\begin{Bmatrix} \vdots \\ \ddot{u}_i \\ \ddot{v}_i \\ \vdots \end{Bmatrix} - M\begin{bmatrix} \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{bmatrix}\begin{Bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \end{Bmatrix} = -M\ddot{U} - MI\begin{Bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \end{Bmatrix} \tag{4-1}$$

where

$$U = \begin{Bmatrix} \vdots \\ u_i \\ v_i \\ \vdots \end{Bmatrix}, \quad M = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}, \quad I = \begin{bmatrix} \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{bmatrix} \tag{4-2}$$

Equilibrium condition of the structure under earthquake ground motion is:

$$\underbrace{C\dot{U}}_{\text{Damping force}} + \underbrace{KU}_{\text{Restoring force}} = \underbrace{-M\ddot{U} - MI\begin{Bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \end{Bmatrix}}_{\text{Inertia force}}$$

Finally the equation of motion is obtained as:

$$M\ddot{U} + C\dot{U} + KU = -MI\begin{Bmatrix} \ddot{X}_0 \\ \ddot{Y}_0 \end{Bmatrix} = R \tag{4-3}$$

2) Numerical integration by Newmark-β method

The incremental formulation for the equation of motion of a structural system is,

$$[M]\{\Delta a_i\}+[C]\{\Delta v_i\}+[K]\{\Delta d_i\}=\{\Delta p_i\} \qquad (4\text{-}4)$$

where, $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices. $\{\Delta d_i\}$, $\{\Delta v_i\}$, $\{\Delta a_i\}$ and $\{\Delta p_i\}$ are the increments of the displacement, velocity, acceleration and external force vectors, that is,

$$\{\Delta d_i\}\equiv\{d_{i+1}\}-\{d_i\},\ \{\Delta v_i\}\equiv\{v_{i+1}\}-\{v_i\},$$

$$\{\Delta a_i\}\equiv\{a_{i+1}\}-\{a_i\},\ \{\Delta p_i\}\equiv\{p_{i+1}\}-\{p_i\} \qquad (4\text{-}5)$$

Using the Newmark-β method,

$$\{\Delta v_i\}=\{a_i\}(\Delta t)+\frac{1}{2}\{\Delta a_i\}(\Delta t) \qquad (4\text{-}6)$$

$$\{\Delta d_i\}=\{v_i\}(\Delta t)+\frac{1}{2}\{a_i\}(\Delta t)^2+\beta\{\Delta a_i\}(\Delta t)^2 \qquad (4\text{-}7)$$

From Equation (4-7), we obtain

$$\{\Delta a_i\}=\frac{1}{\beta(\Delta t)^2}\{\Delta d_i\}-\frac{1}{\beta(\Delta t)}\{v_i\}-\frac{1}{2\beta}\{a_i\} \qquad (4\text{-}8)$$

Substituting Equation (4-7) into Equation (4-6) gives

$$\{\Delta v_i\}=\frac{1}{2\beta(\Delta t)}\{\Delta d_i\}-\frac{1}{2\beta}\{v_i\}+\left(1-\frac{1}{4\beta}\right)\{a_i\}(\Delta t) \qquad (4\text{-}9)$$

Equations (4-8) and (4-9) are substituted into Equation (4-4), and we obtain

$$\{\Delta d_i\}\left[\frac{1}{\beta(\Delta t)^2}[M]+\frac{1}{2\beta(\Delta t)}[C]+[K]\right]$$
$$=\{\Delta p_i\}+[M]\left(\frac{1}{\beta(\Delta t)}\{v_i\}+\frac{1}{2\beta}\{a_i\}\right)+[C]\left[\frac{1}{2\beta}\{v_i\}+\left(\frac{1}{4\beta}-1\right)\{a_i\}(\Delta t)\right] \qquad (4\text{-}10)$$

The equation can be rewritten as,

$$[\hat{K}]\cdot\{\Delta d_i\}=\{\Delta\hat{p}_i\} \qquad (4\text{-}11)$$

where,

$$[\hat{K}]=[K]+\frac{1}{2\beta(\Delta t)}[C]+\frac{1}{\beta(\Delta t)^2}[M] \qquad (4\text{-}12)$$

$$\{\Delta\hat{p}_i\}=\{\Delta p_i\}+[M]\left(\frac{1}{\beta(\Delta t)}\{v_i\}+\frac{1}{2\beta}\{a_i\}\right)+[C]\left[\frac{1}{2\beta}\{v_i\}+\left(\frac{1}{4\beta}-1\right)\{a_i\}(\Delta t)\right] \qquad (4\text{-}13)$$